Representation in Collective Policymaking

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Abstract

We study a policy-motivated principal choosing a representative to bargain over one-dimensional policy. The principal can constrain extremist proposers by (i) shifting (de facto) veto players, or (ii) biasing towards veto players, which improves their policymaking expectations and narrows what would pass. The principal may want to bias inward, but never outward. For a wide interval of principals, optimal representatives are unique, strictly increasing in the principal's ideal point, and biased away from their ideal point towards a central location. In extensions, we study mass representation, the value of representation, and competitive representation.

Keywords: representation, collective policymaking, bargaining, spatial policy, moderation

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Representation in collective policymaking is common but complex. Many policy decisions are made in collective bodies by representatives, such as legislators representing their constituents, committee members representing their party leaders, or judges who politicians appoint to multi-member courts. Since those policies can depend on the overall composition of representatives, as well as their institutional rights and roles (Romer and Rosenthal 1978; Baron and Ferejohn 1989; Krehbiel 1998), the impact of individual representatives can be subtle (Miller and Stokes 1963; Eulau and Karps 1977).[1]

We study the impact and appeal of individual representatives in collective bodies. How do individual representatives impact collective policymaking? Which kinds of representatives are optimal? How does one's optimal representative depend on the characteristics of other members of the collective body or their institutional rights?

We aim to sharpen theoretical understanding by accounting for two key complications of collective policymaking. First, it is interdependent, as individual representatives may impact each other in a variety of ways (Harstad 2010; Gailmard and Hammond 2011). Second, it is uncertain, as forecasts about things like the duration, proposals, or outcomes are typically noisy (Fowler 2006). These two complications can have common sources—e.g., voting rules, procedural rights, agenda congestion, ideological heterogeneity, or polarization—and also impact each other. Although these features and their connection have been incorporated into models of collective policymaking (e.g., Baron and Ferejohn 1987, 1989; Baron 1996; Banks and Duggan 2000, 2006), their consequences for representation are undeveloped.

We analyze a policy-motivated principal choosing the ideal point of their representative, who will bargain with other politicians over one-dimensional policy under simple majority rule. We highlight how different representatives not only behave differently but also induce some of the other politicians to behave differently. The representative's expected behavior impacts which policies can pass and, in turn, affects proposals by extreme politicians. We show that a broad range of principals want to bias their representative inwards, to improve expectations of (de facto) veto players and further constrain extremists. Thus, we find a widespread aversion against more extreme representatives.

Specifically, our collective policymaking setting consists of sequential bargaining over an infinite horizon à la Banks and Duggan (2000). In each period until agreement, a politician is recognized to propose a policy from a one-dimensional policy space and then a simple

[1]Miller and Stokes (1963) highlights that "[t]he legislator acts in a complex institutional setting in which he is subject to a wide variety of influences" (pg. 51) and Eulau and Karps (1977) echoes that "[...] representatives are influenced in their conduct by many forces or pressures or linkages [...]" (pg. 235). More broadly, Pitkin (1967) states that "representation is not any single action by any one participant, but the overall structure and functioning of the system, the patterns emerging from the multiple activities of many people" (pg. 221).

majoritarian vote determines whether their proposal passes or bargaining continues. All players are policy-motivated, with preferences over policy represented by quadratic loss in a *bad status quo* setting where any agreement is preferable to the status quo. The key heterogeneity between politicians is in their ideal points, but we also allow them to have different proposal rights (i.e., recognition probabilities). We fix those proposal rights, however, so that the principal can only choose the representative's ideal point. Accordingly, we isolate the impacts of individual ideological differences between representatives.

Once a representative is in place, equilibrium policymaking induces a unique lottery over policy (Cho and Duggan 2003; Cardona and Ponsati 2011). Whoever proposes first will pass their favorite policy in the set that a majority would pass (Banks and Duggan 2000). Furthermore, that set always coincides with the median politician's *acceptance set* (Duggan 2014), which is an interval of policies around her ideal point. Crucially, it is determined by her expectations about further policymaking—since policymaking can continue after rejected proposals—and thus depends on the profile of politician ideal points and their proposal rights.

The representative's ideology can indirectly affect what some of the other politicians propose in equilibrium. It does so by changing the acceptance set, through shifting either the median's (i) location *or* (ii) policy expectations. The location channel is familiar (Gailmard and Hammond 2011; Klumpp 2010), but the policy expectations channel is less understood. In our setting, the latter is especially pervasive. Regardless of whether the representative would be the median, his mere presence affects the median's willingness to reject proposals due to the prospect that he might subsequently propose. Essentially, the representative's ideology can have *anticipation effects* (Friedrich 1937; Simon 1953). We parse these two channels and characterize how the acceptance set varies with the representative's ideal point. Over representatives who would be the median, it shifts monotonically with the representative's ideal point but its radius can change in different ways, depending on the distribution of proposal rights. Over representatives who would not be the median but are sufficiently moderate that the median accepts the representative's ideal policy if proposed, the acceptance set contracts as the representative shifts towards the median since the median's continuation value from rejecting improves. The representative's ideology has no marginal effect on the distribution over policy outcomes only if the representative is sufficiently extreme such that the representative's ideal policy is rejected by the median if proposed.

Due to the representative's indirect impact on other politicians, the principal faces a classic tradeoff: although a biased representative may propose less favorable policy, they may also induce others to propose more favorable policies (Schelling 1956). We highlight how this general tradeoff may be more widespread than previously appreciated, as it can arise from pervasive anticipation effects. Specifically, we characterize general properties of optimal representatives under minimal assumptions on legislator ideal points and recognition probabilities and show that the principal is always more inclined towards moderation than extremism. Broadly, the principal never strictly prefers someone more extreme and instead always wants someone who is the median or biased in that direction. More precisely, if the principal is in a centrally located interval, then their optimal representative(s) will be the median but can be biased in either direction. Next, if the principal is in either of two intermediate intervals flanking that centrist interval, then their optimal representative(s) are biased strictly inwards, potentially enough to be the median. For any such principal, the downside of biasing their representative's proposal is outweighed by the upside of inducing extremists to further moderate their proposals. Finally, for the remaining (sufficiently extreme) set of principal ideal points, there is always an optimal representative who is more moderate than the principal but their set of optimal representatives never includes a representative who would be the median. Additionally, optimal representatives are ordered. Thus, the principals who want a median representative are an interval containing the medians along with some non-medians on at least one side. Surrounding that interval are two intervals of principals who each want a representative who is more centrist but not a median.

We consider several special cases and extensions of our general model to refine our result and illustrate its potential applications. We first consider a setting in which the other members of the collective body consist of two moderates (e.g., a center-left and center-right party) and two extremists from opposite ends of the policy spectrum. We show that all sufficiently moderate principals bias inward toward a unique central location which is characterized by the balance of extremist proposal rights. That location, the *locus of attraction*, characterizes the unique principal who strictly prefers to have an unbiased representative.[2] Additionally, there is always a *dead zone* of median and non-median representatives who are not optimal for any principal. Furthermore, optimal representatives vary with the balance of extremist proposal rights: principals bias further away from the gaining side to further constrain those extremists.

Next we study a refinement of this special case in which the two moderates have the same ideal point. In this setting, the median is, essentially, fixed so the representative can only affect policymaking through the median's expectations about policy outcomes in future rounds of bargaining. We show that if proposal rights of left- and right-extremists are equal, then the principal's optimal representatives admits a simple closed-form expression equivalent to a convex combination of the median's ideal point and the principal's where the weight placed on her own depends negatively on the median's discount factor and the cumulative recognition probability of extremist legislators. We then use this tractable

[2] An unbiased representative is weakly optimal for very extreme principals.

expression to study the principal's welfare gain from optimal representation — i.e., the *value of representation* — and show that this value is greatest for principals around each moderate-extremist boundary. We then use the fixed-median setting to study multiple principals choosing their own representative's ideology — i.e., *competitive representation.* Specifically, we extend our baseline setting to have two open positions and analyze two opposing principals who simultaneously choose their representative's ideal point. In equilibrium, both principals moderate their representative towards the median but, depending on the balance of extremist proposal rights, they may moderate more or less than in the baseline setting under analogous conditions.

Finally, we study collective choice over the representative's ideal point — i.e., *mass representation* — by extending our general case to allow for the representative to be chosen by a group of principals. We show that preferences over representatives satisfy a single-crossing condition as long as extremist proposal rights are not too high. Under this condition, our results imply that in any pairwise comparison of two arbitrary representatives, the collectively chosen representative corresponds to the representative preferred by a decisive principal (e.g., the median principal under simple majority rule) and that, moreover, the set of ideal points of principals supporting the rightmost representative is always to the right of the set of principals who support the alternative.

Contributions to the Literature

Our results provide insight into representation across various collective policymaking contexts. Our model of collective policymaking is a *minimal legislative process* (Baron 1994) with several interpretations.[3] For instance, it provides a lens for studying representation in separation-of-powers systems[4] (Epstein and O'Halloran 2001; Volden 2002) or, more narrowly, congressional committees.[5] Broadly, we emphasize the role of *ideological* factors for representation, complementing related work emphasizing *distributive* or *informational* factors.[6]

[3]For discussion of interpretations and applications of our bargaining environment, see, e.g., Baron and Ferejohn (1989); Baron (1991); McCarty (2000); Kalandrakis (2006); Eraslan and Evdokimov (2019).

[4]In this vein, we add to Gailmard and Hammond (2011) and Epstein and O'Halloran (2001), who suggest "that theories of legislative organization should be brought out of the legislature and seen as part of our larger constitutional system of policy-making" (pg. 391).

[5]For an overview of scholarship in committee composition, see Evans (2011). Theoretical work on committees has studied, e.g., their *representativeness* (Krehbiel 1990; Hall and Grofman 1990; Cox and McCubbins 2007), who serves on them (Rohde and Shepsle 1973), and the role of intercameral considerations (Diermeier and Myerson 1999; Gailmard and Hammond 2011).

[6]Echoing Fenno (1974), Epstein and O'Halloran (2001) claim that "each of the distributive, informational, and partisan theories predicts outcomes accurately in its own relevant domain; [...] so alternative explanations should be seen as complements rather than substitutes" (pg. 391).

We shed new light on how biased representatives can provide a useful form of commitment (Schelling 1956; Sobel 1981) to improve *other* politicians' behavior enough to outweigh their *own* less-favorable behavior (e.g., Harstad 2010; Christiansen 2013; Loeper 2017).[7] One prominent mechanism is that a status-quo-biased representative with veto power will further constrain extreme proposals (Gailmard and Hammond 2011; Klumpp 2010). Our setting includes that mechanism but also features a different mechanism — the representative's effect on expectations about policymaking — that is lurking in well-known models of collective policymaking (e.g., Banks and Duggan 2000). Since both mechanisms may be present in various settings (e.g., Banks and Duggan 2006), our results complement earlier work by showing how this strategic tension does not require the representative to be a veto player nor the status quo to be strategically relevant.

We highlight a new logic for how moderate representatives can be appealing by reducing extremism. This appeal can arise in various aspects of collective policymaking. First, when allocating proposal rights, risk-averse politicians share an aversion to egalitarianism and would rather shift proposal rights towards moderate members — to make extreme proposals *less likely* (Diermeier et al. 2020).[8] In contrast, we fix (possibly unequal) proposal rights and show a widespread preference for relatively centrist representatives — to make extreme proposals *less extreme*.[9] Second, during bargaining that can continue with accepted policy as the new status quo, proposers may opt for a relatively centrist policy that directly increases the median's reservation value in *future* periods and thus constrains their opposition *in the future* (Baron 1996; Buisseret and Bernhardt 2017; Zápal 2020). In contrast, in our setting a more centrist representative increases the median's reservation value *today*, thereby constraining what extremists can pass *today*. Furthermore, in our analysis, moderate principals want to constrain extremists on both sides, not just their opponents. Third, interest groups seeking access may prefer to target more extreme representatives in order to increase the chances of moderating their proposals, thereby also improving centrist expectations and constraining what extremists can pass (Judd 2023). Our results highlight that beforehand, when the representatives are chosen, similar incentives encourage those groups to support the selection of more moderate candidates.

[7]Also see, e.g., Persson and Tabellini (1992); Besley and Coate (2003). For a general overview of strategic pre-commitment in bargaining, see Miettinen (2022). For related strategic delegation incentives outside the political context, see Dixit (1980); Vickers (1985); Bulow et al. (1985) and Fershtman et al. (1991).

[8]For other related work on *endogenous procedures*, see Diermeier and Vlaicu (2011); Diermeier et al. (2015, 2016). In a dynamic setting with endogenous status quo, Duggan and Kalandrakis (2012) endogenize proposal rights but only study equilibrium existence. In a setting with distributive policy, Eguia and Shepsle (2015) endogenize the set of politicians and their proposal rights.

[9]Two other differences, motivated by our representative/delegate application, are that in our analysis (i) the location of the median policymaker can shift and (ii) we vary the principal's ideal point.

The moderation incentives we uncover also complement extremism incentives driven by collective policymaking in other settings. In a supermajoritarian take-it-or-leave-it setting, voters never want someone more moderate but may prefer a strictly more extreme representative who would be a veto pivot (Kang 2017). In other contexts where policy is a weighted average of politician ideal points, extreme representatives can counterbalance extreme opponents (Alesina and Rosenthal 1996; Kedar 2005, 2009). Additionally, if principals care about how their representative will vote on an *exogenous* legislative agenda, then preferences can be asymmetric and favor extremism (Patty and Penn 2019). Understanding these different directions can help inform empirical interpretation and anticipation of future choices.[10]

Model

Players. There is a principal, P; a continuum of potential representatives; and a set of k (even) auxiliary politicians, K.

Timing. The model has two stages. First, in the *appointment* stage, P selects a representative, denoted d, to bargain on her behalf. Second, in the *bargaining* stage, the representative d interacts with the other politicians in K to collectively set a one-dimensional policy. Each bargaining period $t \in \{1, 2, ...\}$, a politician $i \in N = K \cup \{d\}$ is drawn from the recognition distribution ρ, where $\rho_i \in (0, 1)$ for all i and $\sum_{i \in N} \rho_i = 1$, and then i proposes a policy $x^t \in X = [0, 1]$. Next, all politicians vote to accept or reject x^t. The proposal is approved if and only if a simple majority of individuals approve. If x^t is approved, then it is implemented and the game ends. Otherwise, the proposal is rejected and the game moves to $t + 1$. Bargaining continues indefinitely until a proposal is accepted.

Preferences. All players are purely policy-motivated and each player has a unique ideal point $y_i \in X$. We denote the principal's ideal point as y_p and the ideal point of her chosen representative as y_d. The ideal points of the k legislators in K are ordered such that $y_1 \leq y_2 \leq ... \leq y_k$, and we denote $\ell = \frac{k}{2}$ and $r = \frac{k}{2} + 1$. The median legislator in N depends on y_d and is denoted

$$m = \begin{cases} \ell & \text{if} \quad y_d < y_\ell \\ d & \text{if} \quad y_d \in [y_\ell, y_r] \\ r & \text{if} \quad y_d > y_r. \end{cases} \tag{1}$$

Once a policy x is enacted, player i will receive policy utility $u(x, y_i) = 1 - (x - y_i)^2 \geq 0$

[10] Extremism can also emerge if the principal does not know their appointee's ideology but does know they will serve on a collective body that sets policy at the median ideal point (Bailey and Spitzer 2018).

each period thereafter. Before then, every player receives zero utility in each period until agreement.

Cumulative payoffs are sums of per-period utilities, discounted by the common factor $\delta \in (0, 1)$. We normalize per-period utility by the factor $1 - \delta$. Thus, if x is accepted in period t, then legislator i's payoff is $\delta^{t-1}u(x, y_i)$.

Information. All features of the game are common knowledge.

Strategies & Equilibrium concept. In the appointment stage, a pure strategy for the principal prescribes a choice of d's ideal point, $y_d \in X$. We focus on a standard class of bargaining strategies (Banks and Duggan 2000; Cardona and Ponsati 2011) that are relatively simple and focal (Baron and Kalai 1993; Baron 1994), with politicians always voting as if pivotal (Duggan and Fey 2006). In the bargaining stage, a pure *stationary strategy* for each individual $i \in N$ prescribes (i) a proposal, x_i, that he makes at any t he is selected to propose; and (ii) an acceptance set, A_i, that specifies a time-independent set of proposals that he accepts or rejects. A *stationary subgame perfect equilibrium* in the bargaining subgame is a profile of stationary strategies that are mutual best responses in each subgame of the bargaining subgame. An *equilibrium* is a strategy profile in which (i) players in the bargaining subgame play stationary subgame perfect equilibrium strategies and (ii) P chooses y_d to maximize her expected payoff anticipating the distribution of policy outcomes that y_d will induce.

Analysis

We first characterize equilibrium behavior during the bargaining stage, after y_d is chosen. Then, we trace how y_d affects d's behavior, as well as that of other politicians. Next, we study the principal's preference over y_d and how her set of optimal representatives varies with her ideology. Finally, we study several special cases and extensions.

Equilibrium policymaking

Lemma 1 summarizes properties of equilibrium policymaking. Figure 1 illustrates.

Lemma 1. *For each $y_d \in X$, the following hold:*

1. *(Banks and Duggan 2000) An equilibrium exists and it is a no-delay pure strategy equilibrium.*

2. *(Cho and Duggan 2003; Cardona and Ponsati 2011) There is a unique equilibrium acceptance set, $A(y_d)$, of proposals that are approved if proposed and each politician i*

proposes the policy $x \in A(y_d)$ that minimizes $|x - y_i|$. The equilibrium acceptance set is an interval, $A(y_d) = [\underline{x}(y_d), \overline{x}(y_d)]$ and is equivalent to the set of proposals accepted by the median legislator.

3. *A $\underline{\delta}_{y_d} < 1$ exists such that $A(y_d) \subset (0, 1)$ if and only if $\delta > \underline{\delta}_{y_d}$.*

Moreover, $A(y_d)$ is continuous in y_d, δ, and ρ.

In a stationary equilibrium, there is a social acceptance set of proposals $A = [\underline{x}, \overline{x}]$ that are approved in each round of bargaining. Each legislator i's best response to A is to propose $\text{argmin}_{x \in A} |y_i - x|$. Given that all legislators play this proposal strategy, legislator i's expected payoff at the beginning of each round of bargaining—i.e., his *continuation value*—is

$$V_i(A) = P(\underline{x})u(\underline{x}, y_i) + (1 - P(\overline{x}))u(\overline{x}, y_i) + \sum_{j \in N: y_j \in (\underline{x}, \overline{x}]} \rho_j u(y_j, y_i), \tag{2}$$

where $P(x) \equiv \sum_{i \in N: y_i \leq x} \rho_i$ denotes the cumulative proposal rights of politicians left of x. Each legislator's voting strategy in response to an arbitrary A and all legislators proposing $\text{argmin}_{x \in A} |y_i - x|$ is characterized by an individual acceptance set,

$$A_i(A) = \{x \in X | u(x, y_i) \geq \delta V_i(A)\}, \tag{3}$$

of proposals that i approves in each round of bargaining. A proposal is accepted if a coalition of $(n + 1)/2$ approve it so the social acceptance set is a fixed point of

$$A = \bigcup_{C \in 2^N : |C| \geq \frac{n+1}{2}} \bigcap_{i \in C} A_i(A).$$

In equilibrium, this set is uniquely determined by the set of proposals that the median legislator accepts. The equilibrium acceptance set, $A(y_d) = [\underline{x}(y_d), \overline{x}(y_d)]$, is therefore characterized by the unique fixed point of $A_m(A) = \{x \in X | u(y_m, x) \geq \delta V_m(A)\}$ (Cardona and Ponsati 2011).

Since $u(\cdot, \cdot)$ is quadratic, the acceptance set admits a simple characterization when $A(y_d) \subset (0, 1)$, namely, an interval centered on y_m where $\overline{x}(y_d) = 2y_m - \underline{x}(y_d)$ and

$$\underline{x}(y_d) = x \in (0, y_m) \text{ such that } u(x, y_m) = \delta V_m([\underline{x}(y_d), 2y_m - \underline{x}(y_d)]). \tag{4}$$

To simplify analysis, we focus on cases in which $A(y_d) \subset (0, 1)$ for all y_d by assuming that players are sufficiently patient. We use the third part of Lemma 1, which follows from Banks and Duggan (2000) who show that for a given y_d, the acceptance set converges to $\{y_m\}$ as $\delta \to 1$. Thus $A(y_d) \subset (0, 1)$ for all y_d if $\delta > \underline{\delta} \equiv \max \underline{\delta}_{y_d}$.

Assumption 1. $\delta > \underline{\delta}$.

By Lemma 1, every y_d induces a unique equilibrium policy lottery with mean $\mu(y_d)$ and variance $\sigma^2(y_d)$. The boundaries of $A(y_d)$ and the ideal points in its interior are each weighted by the recognition probability of the politicians who propose them.

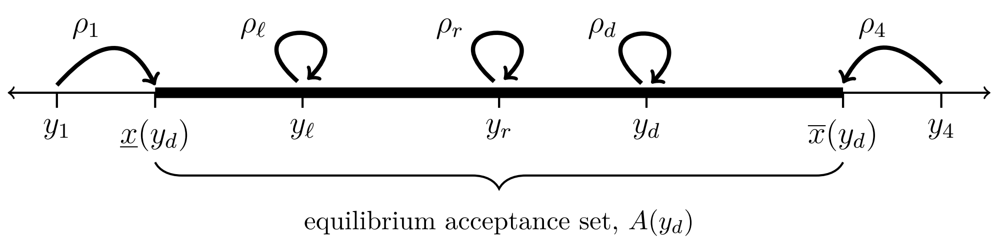
Remark 1. *Given y_d, the unique equilibrium policy lottery puts probability $P(\underline{x}(y_d))$ on $\underline{x}(y_d)$; $1 - P(\overline{x}(y_d))$ on $\overline{x}(y_d)$; ρ_i on each y_i in $(\underline{x}(y_d), \overline{x}(y_d)]$; and zero otherwise.*

Since u is quadratic and each y_d induces a unique lottery over policies, a player i's equilibrium value can be expressed in terms of y_d as

$$V_i(A(y_d)) = 1 - (y_i - \mu(y_d))^2 - \sigma^2(y_d). \tag{5}$$

We use both the weighted-sum expression of V_i from (2) and the mean-variance version from (5) in our analysis below as some results are easier to show with one than the other. In both expressions, Remark 1 highlights how V_i can depend on y_d through d's proposal and $A(y_d)$.

Figure 1: Illustration of equilibrium policymaking (given y_d)



Note: Figure 1 illustrates Lemma 1 for a hypothetical five-member legislature with $y_d > y_r$. The acceptance set is the bold interval, which is centered around $y_m = y_r$. Arrows point from legislators to their proposals (if recognized). Each legislator proposes the closest acceptable policy.

The representative's effects on policymaking

We now study how the representative's ideal point, y_d, affects policymaking. First, we characterize when it affects the representative's proposals. Then, we show how it affects the acceptance set and, in turn, proposals by other politicians.

The representative's proposal varies with y_d if and only if he will not be constrained by the acceptance set — i.e., $y_d \in \text{int } A(y_d)$. We first show that the set of y_d such that $y_d \in \text{int } A(y_d)$ is an open interval containing $[y_\ell, y_r]$.

Lemma 2. *There are unique $\underline{x}_\ell \in (0, y_\ell)$ and $\overline{x}_r \in (y_r, 1)$ such that $y_d \in intA(y_d)$ if and only if $y_d \in (\underline{x}_\ell, \overline{x}_r)$.*

To establish Lemma 2, we consider the median's choice of A in response to d proposing an arbitrary x_d and all $i \in K$ proposing $x_i^* = \text{argmin}_{x \in A} |y_i - x|$. For an arbitrary A, let $\tilde{V}_m(A, x_d) = \sum_{i \in K} \rho_i u(y_m, x_i^*) + \rho_d u(y_m, x_d)$ and $\tilde{A}_m(A, x_d) = \{x \in X | u(y_m, x) \geq \delta \tilde{V}_m(A, x_d)\}$. Note that $\tilde{V}_m(A, x_d)$ is equivalent to the median's equilibrium continuation value if $x_d = x_d^*$ and $A = A(y_d)$; and that $A(y_d)$ is the unique fixed point of $\tilde{A}_m(A, x_d)$ when $x_d = x_d^*$. We use this identify the set of policies, A, that $m = \ell$ accepts if all $i \in K$ propose $x_i^*(A)$ and d proposes $\underline{x}$ (regardless of y_d). Let $A_\ell = [\underline{x}_\ell, \overline{x}_\ell]$ denote a fixed point of $\tilde{A}_\ell(A, \underline{x})$. Since d's proposal only affects $\tilde{V}_\ell(A, \underline{x})$ through $\underline{x}$, A_ℓ is a fixed point of $\tilde{A}_\ell(A, \underline{x})$ for all y_d. Lemma 1 implies that in equilibrium, if $y_d = 0$, then d proposes $\underline{x}(y_d)$, since $\underline{x}(y_d) \geq 0$ for all y_d and $m = \ell$. Since the equilibrium acceptance set is unique, it follows that A_ℓ is unique and that $A(0) = A_\ell$. Analogously, there is a unique set of policies $A_r = [\underline{x}_r, \overline{x}_r]$ that $m = r$ accepts if all $i \in K$ propose $x_i^*(A)$ and d proposes $\overline{x}$, regardless of y_d. This corresponds to the equilibrium acceptance set if $y_d = 1$, i.e., $A(1) = [\underline{x}_r, \overline{x}_r]$. Since d, in equilibrium, proposes $x_d = \underline{x}(y_d)$ if and only if $y_d \leq \underline{x}(y_d)$, proposes $x_d = \overline{x}(y_d)$ if and only if $y_d \geq \overline{x}(y_d)$, and otherwise always proposes $x_d \in [\underline{x}(y_d), \overline{x}(y_d)]$, it follows that $y_d \in \text{int}\, A(y_d)$ if and only if $y_d \in (\underline{x}_\ell, \overline{x}_r)$.

Since $[y_\ell, y_r] \subset (\underline{x}_\ell, \overline{x}_r)$, we can partition the set of potential representatives, X based on whether they would be outside the acceptance set, inside the acceptance set but not the median, orthe median. We label these three cases in Definition 1.

Definition 1. *A player i is **centrist** if $y_i \in [y_\ell, y_r]$; **extremist** if $y_i \notin (\underline{x}_\ell, \overline{x}_r)$; and **moderate** otherwise.*

How $A(y_d)$ varies with y_d depends on whether d is a centrist, moderate, or extremist. These effects depend on how y_d impacts the center of the acceptance set, y_m, or radius, via V_m.

Extremist y_d on each side do not have any marginal impact on $A(y_d)$, since y_m is constant on each interval of extremists and all these representatives propose the same boundary of $A(y_d)$. Hence $A(y_d)$ is constant over each interval of extremists: $A(y_d) = A_\ell$ if $y_d \leq \underline{x}_\ell$ and $A(y_d) = A_r$ if $y_d \geq \overline{x}_r$. In the Appendix, we show that, additionally, $\underline{x}_\ell < \underline{x}_r$ and $\overline{x}_\ell < \overline{x}_r$.

Moderate y_d on each side only impact V_m, since y_m is constant but changes to d's proposal will change V_m. As y_d moves closer to y_m, V_m strictly increases so the acceptance set shrinks as y_d shifts inward over each interval $(\underline{x}_\ell, y_\ell)$ and $(y_r, \overline{x}_r)$. Moreover, the upper and lower bounds change at the same rate. Those changes vanish as y_d approaches the centrists—since u is strictly concave and differentiable, the effect of d's proposal on V_m converges to zero as $y_d \to y_m$.

Centrist y_d impact both the location of y_m, since $y_m = y_d$, and V_m, since y_m shifts relative to the other potential proposers. These two effects can oppose each other but the first always dominates, so the acceptance set shifts to the right as y_d does. Lemma 3 summarizes these observations. Figure 2 illustrates.

Lemma 3. *$A(y_d) = [\underline{x}(y_d), \overline{x}(y_d)]$ is continuous and $A(y_d) \subset [\underline{x}_\ell, \overline{x}_r]$ for all y_d, where:*

1. *$A(y_d) = [\underline{x}_\ell, \overline{x}_\ell]$ for all $y_d \leq \underline{x}_\ell$;*
2. *$A(y_d) \subset [\underline{x}_\ell, \overline{x}_\ell]$ for each $y_d \in (\underline{x}_\ell, y_\ell)$ with $\underline{x}(y_d)$ strictly increasing and $\overline{x}(y_d)$ strictly decreasing at equal rates which converge to zero as $y_d \to y_\ell$;*
3. *$A(y_d) \subset [\underline{x}(y_\ell), \overline{x}(y_r)]$ for each $y_d \in [y_\ell, y_r]$, with $\underline{x}(y_d)$ and $\overline{x}(y_d)$ strictly increasing;*
4. *$A(y_d) \subset [\underline{x}_r, \overline{x}_r]$ for each $y_d \in (y_r, \overline{x}_r)$, with $\underline{x}(y_d)$ strictly decreasing and $\overline{x}(y_d)$ strictly increasing at equal rates which converge to zero as $y_d \to y_r$; and*
5. *$A(y_d) = [\underline{x}_r, \overline{x}_r]$ for all $y_d \geq \overline{x}_r$.*

Moreover, $\underline{x}_\ell \leq \underline{x}_r$ and $\overline{x}_\ell \leq \overline{x}_r$.

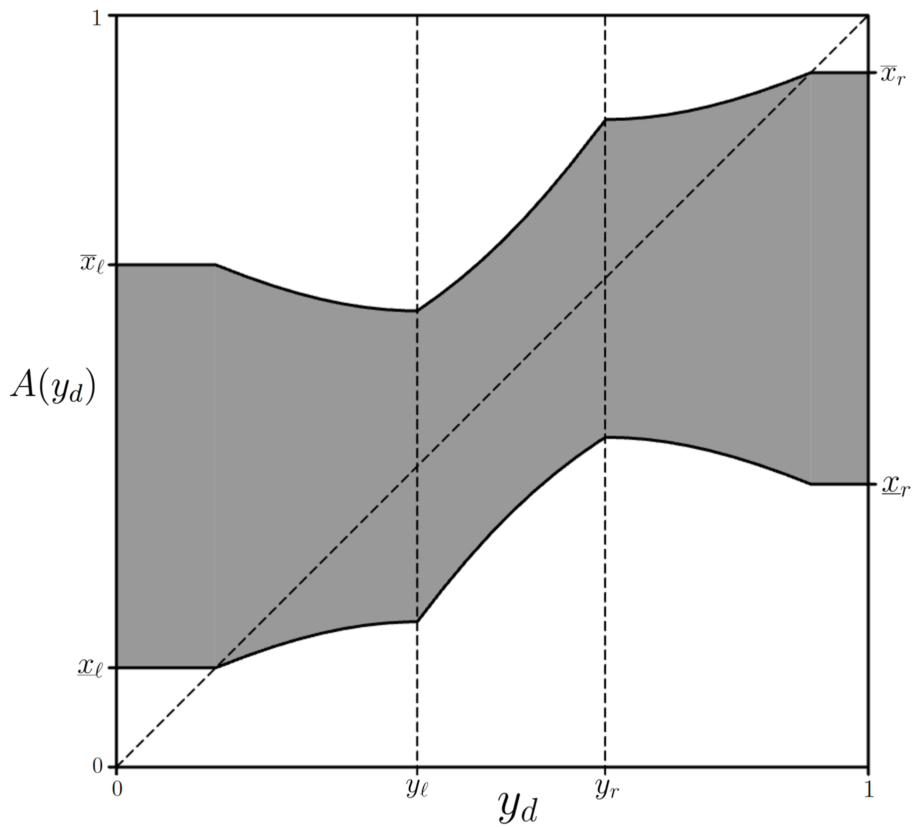
Additionally, $\underline{x}(y_d)$ and $\overline{x}(y_d)$ are differentiable almost everywhere on $(\underline{x}_\ell, \overline{x}_r)$. Left and right derivatives are unequal only at y_ℓ, y_r, and any y_d where the ideal point of some legislator $i \in K$ exits or enters the acceptance set, i.e., at a y_d such that $\underline{x}(y_d) = y_i$ or $\overline{x}(y_d) = y_i$ for some $i \in K$.[11] Moreover, on each interval where $A(y_d)$ is smooth: $\underline{x}(y_d)$ is strictly concave and $\overline{x}(y_d)$ strictly convex since the median's utility is quadratic. At at $y_d = y_\ell$ and $y_d = y_r$ and any y_d where a legislator exits or enters $A(y_d)$, the rate of change of $\underline{x}(y_d)$ discontinuously rises and conversely for $\overline{x}(y_d)$, as Figure 2 illustrates at $y_d = y_\ell$ and $y_d = y_r$.

The principal's expected payoff is determined by y_d through the lottery it induces over policy outcomes. In general the family of probability distributions over $x \in X$ parametrized by $y_d \in X$ is not ordered by y_d. However, there are subsets of X on $\mu(y_d)$ and $\sigma^2(y_d)$ that are ordered in a manner facilitating our characterization of optimal representatives via (5). To state these properties formally, let $\tilde{P}(x) \equiv \sum_{i \in K: y_i \leq x} \rho_i$ denote the cumulative recognition probability of exogenous legislators with ideal points weakly the left of x.

Lemma 4. *There exist $\underline{\pi} \in [\underline{x}_\ell, y_\ell)$ and $\overline{\pi} \in (y_r, \overline{x}_r]$ such that $\mu(y_d)$ is strictly increasing on $[\underline{\pi}, \overline{\pi}]$. Moreover, $\underline{\pi} > \underline{x}_\ell$ implies $1 - \tilde{P}(y_\ell) > \frac{1}{2\delta}$ and $\overline{\pi} = \overline{x}_r$; and $\overline{\pi} < \overline{x}_r$ implies $\tilde{P}(y_\ell) > \frac{1}{2\delta}$ and $\underline{\pi} = \underline{x}_\ell$. Additionally, on any interval $Z \subset [\underline{x}_\ell, \underline{\pi}]$ such that $\mu(y_d)$ is decreasing, $y_\ell < \mu(y_d)$ and $\sigma^2(y_d)$ is strictly decreasing; and on any interval $Z \subset [\overline{\pi}, \overline{x}_r]$ such that $\mu(y_d)$ is decreasing, $\mu(y_d) < y_r$ and $\sigma^2(y_d)$ is strictly increasing.*

[11] The set of y_d that satisfy the latter condition depends on ρ, δ, and $(y_1, ...y_k)$. There at most $2K$ points in this set since $\underline{x}(y_d) < y_r$, is increasing on $y_d \leq y_r$, and decreasing on $y_d \geq y_r$.

Figure 2: How the acceptance set varies with y_d



Note: Figure 2 illustrates how $A(y_d)$ (vertical axis), varies with y_d (horizontal axis), for a five-member legislature where $\delta = .98$, $(\rho_1, ..., \rho_4, \rho_d) = (.2, .15, .2, .18, .27)$, and $(y_1, ..., y_4) = (0, .4, .65, 1)$.

By Lemma 3, right-extremist representatives induce a lottery with a greater mean than left-extremists. On the interval of centrists, both boundaries of $A(y_d)$ are strictly increasing in y_d, so lotteries are ordered by first order stochastic dominance on $[y_\ell, y_r]$. Since the rate of change of the boundaries of $A(y_d)$ converges to zero on each interval of moderates as y_d approaches y_ℓ or y_r, there is an interval $[\underline{\pi}, \overline{\pi}]$—including moderates from both sides and all centrists—on which $\mu(y_d)$ strictly increases. Thus if $\mu(y_d)$ decreases on any interval of X, it only does so locally within the set of either left- or right-moderates. There need not be any such interval, however, and they can only occur on one side because decreasing $\mu(y_d)$ on a subinterval of moderates requires that extremist representatives from the opposite side of m have sufficient proposal rights.

To illustrate, consider a left-moderate, $y_d < y_\ell$. As y_d shifts towards y_ℓ, the lower bound and y_d shift rightward putting upward pressure on μ while the upper bound shifts in, putting downward pressure on μ. Sufficiently high proposal rights for right-extremists (greater than $\frac{1}{2\delta}$) can make μ decrease if the boundaries of $A(y_d)$ contract at a sufficiently fast rate. These conditions imply that $\mu(y_d) > y_\ell$ and that $\sigma^2(y_d)$ decreases. Thus on any interval of left-moderates where $\mu(y_d)$ decreases, shifting y_d to the right pulls $\mu(y_d)$ leftward towards y_ℓ and reduces the variance of the policy lottery. Conditions are analogous for $\mu(y_d)$ to decrease on an interval of right-extremists, requiring that left-extremists have proposal rights exceeding $\frac{1}{2\delta}$. Since $\frac{1}{2\delta} > \frac{1}{2}$, the mean can only decrease on only one side.

Optimal Representation

We apply our understanding of the representative's effect on bargaining to characterize optimal representatives. Since each y_d induces a unique policy lottery, the principal's expected utility given $y_p \in X$ is uniquely defined over $y_d \in X$,

$$U(y_d, y_p) \equiv V_p(A(y_d)).$$

We characterize the optimal representative correspondence $y_d^* : X \rightrightarrows X$, where:

$$y_d^*(y_p) \equiv \operatorname*{argmax}_{y_d \in X} U(y_d, y_p) \tag{6}$$

is non-empty, upper hemicontinuous, and compact-valued since $U(y_d, y_p)$ is continuous in y_d.

Proposition 1. *The optimal representative correspondence y_d^* is increasing in the strong set order sense. Moreover, there are intervals (E_L, E_R) and $(\underline{y}_p, \overline{y}_p)$ satisfying $(E_L, E_R) \supset (\underline{x}_\ell, \overline{x}_r) \supset [\underline{y}_p, \overline{y}_p] \supset [y_\ell, y_r]$ such that y_d^* is single-valued and continuous almost everywhere on (E_L, E_R) and:*

1. *$y_p < E_L$ implies $y_d^*(y_p) = [0, \underline{x}_\ell]$;*
2. *$y_p \in (E_L, \underline{y}_p)$ implies $y_d^*(y_p) \subset (y_p, y_\ell)$;*
3. *$y_p \in (\underline{y}_p, \overline{y}_p)$ implies $y_d^*(y_p) \subset [y_\ell, y_r]$;*
4. *$y_p \in (\overline{y}_p, E_R)$ implies $y_d^*(y_p) \subset (y_r, y_p)$; and*
5. *$y_p > E_R$ implies $y_d^*(y_p) = [\overline{x}_r, 1]$.*

To build intuition for Proposition 1, we start by identifying which representatives are candidates for an optimum depending on whether y_p is an extremist or moderate from a given side, or a centrist. To facilitate analysis, we define, for each y_p, the set of aligned representatives as the set of y_d that are on the same side of X as y_p relative to y_ℓ and y_r.

Definition 2. *Given y_p, a representative y_d is **aligned** if either $y_\ell \leq \min\{y_p, y_d\}$ or $\max\{y_p, y_d\} \leq y_r$. Otherwise, y_d is **opposed**.*

Note that only centrist representatives are aligned for centrist principals. For a moderate or extremist principal, the set of aligned representatives consists of all centrists and all moderate and extremist representatives from the same side of X as y_p.

It follows from Proposition 1 that any optimal representative must be aligned with the principal. This rules out any $y_p \leq y_r$ choosing $y_d > y_r$, or $y_p \geq y_\ell$ choosing $y_d < y_\ell$. Throughout our analysis, we present intuition for $y_p \leq y_r$ cases since results for right-moderate and right-extremist principals are analogous. For $y_p \leq y_r$, shifting y_d to the right of y_r pushes d's proposal and the proposals of right-extremists away from y_p so any benefit from shifting rightward comes from pulling $\underline{x}(y_d)$ closer to y_p. But shifting y_d in either direction away from y_r pulls $\underline{x}(y_d)$ to the left. Thus any principal who benefits from shifting y_d to the right of y_r can achieve the same benefit by instead shifting y_d leftward but without the cost of shifting $\overline{x}(y_d)$ further away.

It immediately follows that centrist principals always choose a centrist representative.

A moderate principal chooses either an aligned moderate or centrist but never an extremist. Moreover, her chosen representative is never as extreme or more extreme than she is. To see why, consider $y_p \in (\underline{x}_\ell, y_\ell)$. If the principal chooses $y_d = y_p$, then $y_p \in A(y_d)$. Shifting y_d to the left pushes d's proposal and the proposals of extremists further away, so $y_d < y_p$ is never optimal. Shifting y_d rightward, on the other hand, pulls both boundaries of the acceptance set closer to y_p at the cost of pushing d's proposal further away. At $y_d = y_p$, the marginal cost of constraining extremist proposals in terms of a worse proposal by d is zero, so $y_p < \min y_d^*(y_p)$.

An extremist principal chooses either an aligned extremist or an aligned moderate. If the principal chooses an aligned extremist, then because U is constant over each set of extremist representatives, she is indifferent between all aligned extremists, e.g., for a left-extremist, $[0, \underline{x}_\ell] \subset y_d^*(y_p)$ if $[0, \underline{x}_\ell] \cap y_d^*(y_p) \neq \emptyset$. Since, extremists are never in the acceptance set by definition, if the principal is a left-extremist, then $y_p < \underline{x}(y_d)$ for all y_d. Because both boundaries of the $A(y_d)$ shift rightward with y_d on $[y_\ell, y_r]$, a centrist representative is never optimal. On the other hand, starting at $y_d = \underline{x}_\ell$ and shifting y_d rightward into the set of left-moderates pushes d's proposal and aligned extremist proposals away from y_p but pulls opposing extremist proposals closer. If y_p is sufficiently close to $\underline{x}_\ell$, the marginal cost of constraining opposing extremists is near zero so there is always a non-empty interval of extremists who choose an aligned moderate. Indeed, if any $y_p < \underline{x}_\ell$ chooses an extremist, then because $y_p = \underline{x}_\ell$ chooses $y_d > \underline{x}_\ell$, the upper hemicontinuity of y_d^* implies that there exists a unique extremist $e_L \in (0, \underline{x}_\ell)$ such that $y_d^*(e_L) = [0, \underline{x}_\ell]$ and $\underline{x}_\ell < \min y_d^*(y_p)$ for all

$y_p \in (e_L, \underline{x}_\ell)$. While there are always extremists who choose a moderate, it is possible for no extremists to choose an extremist. Let $E_L = 0$ if all left-extremists choose $y_d > \underline{x}_\ell$ and let $E_L = e_L$ otherwise. Define $E_R > \overline{x}_r$ analogously.

Note that we have not yet ruled out some $y_p < E_L$ or $y_p > E_R$ choosing a non-extremist. Our next step is to show that y_d^* is ordered on X in the strong set order sense. In general, $U(y_d, y_p)$ is not guaranteed to satisfy the single-crossing property on X. Quadratic loss implies that for any $Y \subseteq X$, U satisfies single-crossing on Y if $\mu(y_d)$ is weakly increasing in y_d and strict single-crossing (Milgrom and Shannon 1994) if $\mu(y_d)$ is strictly increasing.[12] Using Lemma 4, we can show that $\mu(y_d)$ is increasing on the image of y_d^*, i.e. $\tilde{Y} = \{y_d^*(y_p) | y_p \in X\}$. Since $y_d^*(y_p) = \text{argmax}_{y_d \in \tilde{Y}} U(y_d, y_p)$ and U satisfies single crossing on $\tilde{Y}$, this is sufficient for $y_d^*(y_p)$ to be increasing in the strong set order sense.

To establish that $\mu(y_d)$ is increasing on $\tilde{Y}$, first note that by Lemma 4, μ can decrease only on an interval of moderates on one side. If such an interval exists and is within the set of left-moderate representatives, then because the principal's optimal representatives must be aligned, only a left-extremist or left-moderate principal could potentially choose a representative from this interval. However, it follows from Lemma 4 that for $\mu(y_d)$ to decrease as y_d moves closer to the median, y_ℓ, the proposal rights for right-extremists must be sufficiently high such that $y_\ell < \mu(y_d)$. Thus shifting y_d to the right pulls $\mu(y_d)$ closer to all $y_p < y_\ell$. Since shifting y_d rightward towards the median also reduces the variance of the policy distribution, $U(y_d, y_p)$ is strictly increasing for all $y_p < y_\ell$ on any interval where $\mu(y_d)$ is decreasing. Therefore $\mu(y_d)$ is weakly increasing locally at every $y_d \in \tilde{Y}$.

Additionally, we can rule out the principal choosing one left-moderate, y_d, over another $y_d' \in (y_d, y_\ell)$ such that $\mu(y_d') = \mu(y_d)$. Since V_ℓ is strictly increasing in y_d on the interval of left-moderate representatives, $\mu(y_d') = \mu(y_d)$ and $y_d < y_d'$ imply that $\sigma^2(y_d') < \sigma^2(y_d)$ via (5). Thus all principals strictly prefer y_d'. An analogous result holds for right-moderate representatives where all principals prefer the representative closer to y_r if two right-moderate representatives induce distributions with equal means. It therefore follows that $\mu(y_d)$ is weakly increasing in y_d on the image of y_d^* and, additionally, strictly increasing on the image of y_d^* under (E_L, E_R), i.e., $\{y_d^*(y_p) | y_p \in (E_L, E_R)\}$. Thus y_d^* is increasing in the strong set order sense on X. Moreover, every selection from $y_d^*|_{(E_L, E_R)}$ is increasing and, consequently, y_d^* is a singleton almost everywhere on (E_L, E_R).

Viewed from another angle, Proposition 1 also characterizes *which* types of principal would want a centrist, moderate, or extremist representative. Those preferring a centrist are in a centrally-located interval, $(\underline{y}_p, \overline{y}_p)$, including some moderates, potentially on both sides.

[12]This is because (strictly) increasing $\mu(y_d)$ is implies that U satisfies (strict) increasing differences and thus (strict) single crossing.

Those who prefer a moderate are on either side of that interval, in two intermediate intervals, $(E_L, \underline{y}_p)$ and $(\overline{y}_p, E_R)$, each including some extremists. Those who prefer an extremist are even further out, in two intervals, $[0, E_L)$ and $(E_R, 1]$, containing only extremists (and potentially empty).

Discontinuities in y_d^* on (E_L, E_R) can occur due to the fact that at each y_d such that a $y_j \in K$ enters or exits $A(y_d)$, the rate of change of the boundaries of the acceptance set jumps up or down. At either $y_p = \underline{y}_p$ or $y_p = \overline{y}_p$ (or both) $y_d^*(y_p)$ may be multivalued, containing a moderate and a centrist representative.[13] To understand the source of these potential discontinuities and characterize $(\underline{y}_p, \overline{y}_p)$, recall from Lemma 3 that as y_d approaches y_ℓ, the rate at which the boundaries of $A(y_d)$ change goes to zero. Thus for a left-moderate principal, shifting y_d arbitrarily close to y_ℓ from the left provides a near-zero marginal benefit in terms of constraining extremist proposals at a constant and positive marginal cost of pushing d's proposal further away. It is therefore never optimal for a left-moderate to choose y_ℓ. To constrain extremist proposals—aligned extremists in particular—it is more efficient for left-moderates near y_ℓ to choose a centrist. Thus if the proposal rights for left-extremists are sufficiently high such that a principal with $y_p = y_\ell$ wants to shift her representative rightward into the set of centrists, principals near the moderate-centrist boundary want to bias inward too in order to constrain aligned extremist proposers, with $y_p = \underline{y}_p < y_\ell$ uniquely indifferent between a strict centrist or strict moderate. Otherwise, all left-moderate principals choose a left-moderate representative, so $\underline{y}_p = y_\ell$, and $y_d^*(y_p) \to \{y_\ell\}$ as $y_p \to y_\ell$.

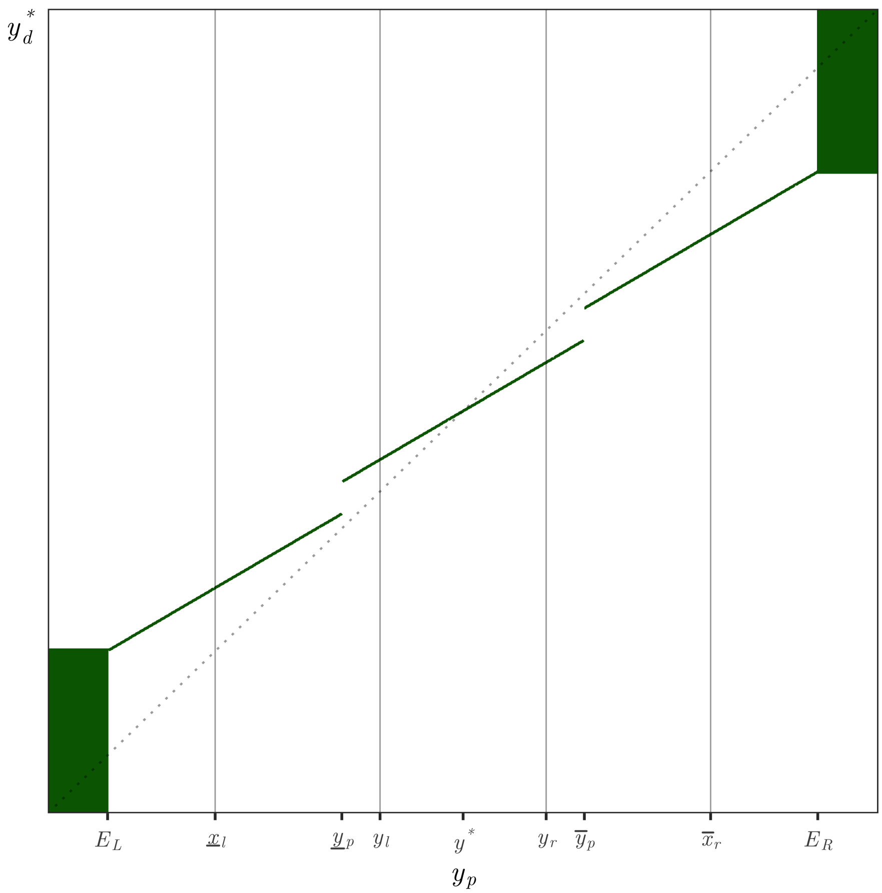
Finally, our results show which principals can self-represent, i.e., $y_p \in y_d^*(y_p)$. Clearly all $y_p \notin [E_L, E_R]$ can self-represent since every aligned extremist is optimal. Within (E_L, E_R), only centrists can self-represent and there is at least one centrist who does.

Corollary 1. *There exists a non-empty finite set $Y^* \subset [y_\ell, y_r]$ such that if $y_p \in (E_L, E_R)$, then $y_p \in y_d^*(y_p)$ if and only if $y_p \in Y^*$. Moreover, $y_d^* \subset (y_p, \min Y^*)$ for all $y_p \in (E_L, \min Y^*)$ and $y_d^* \subset (\max Y^*, E_R)$ for all $y_p \in (\max Y^*, E_R)$.*

Corollary 1 refines our characterization of optimal representatives on (E_L, E_R). There is a unique centrist representative, $y_d = \min Y^*$ towards which all leftward principals strictly shift and, analogously another unique centrist, $y_d = \max Y^*$, that all rightward principals strictly shift their representative towards.

[13] At all other discontinuities, y_d^* contains only centrists or only moderates, though such discontinuities may not exist as our example below in Figure 1 illustrates.

Figure 3: Optimal representatives



Note: Figure 3 illustrates key properties of the optimal representative characterization in Proposition 1 for a five-member legislature where $\delta = .99$, $(\rho_1, \rho_4, ...\rho_d) = (.2, .25, .25, .2, .1)$, and $(y_1, ..., y_4) = (0, .4, .6, 1)$: (i) very extreme principals, $y_p \notin (E_L, E_R)$, prefer any aligned extremist; (ii) intermediate principals, $y_p \in (E_L, \underline{y}_p) \cup (\overline{y}_p, E_R)$ prefer a unique moderate who is strictly more centrist; and (iii) centrist principals, $y_p \in (\underline{y}_p, \overline{y}_p)$, prefer a unique centrist.

Applications

Polarized Legislature

To illustrate our result and study the model's implications in a tractable setting, we consider a 5-member legislature with two exogenous legislators, ℓ and r, whose ideal points are always in $A(y_d)$ and two others, which we label L and R, whose ideal points are $y_L = 0$ and $y_R = 1$. Under Assumption 1, y_L and y_R are always outside of $A(y_d)$. To ensure $y_\ell, y_r \in A(y_d)$ for all y_d under Assumption 1, it is sufficient to assume that y_r is sufficiently close to y_ℓ.[14] This

[14] The explicit bound in Assumption 2 follows from Lemma 3.1 in Predtetchinski (2011).

setting represents a legislature with two extreme factions on each side and two moderate factions, e.g., a two-party legislature, where each party contains a moderate and extreme caucus. We formalize the assumptions for this special case in Assumption 2.

Assumption 2 (Polarized Legislature)**.** *There are four auxiliary legislators in K with ideal points $0 = y_L < y_\ell \leq y_r < y_R = 1$ and $(y_r - y_\ell) < (1-\delta)/2$.*

This setting provides additional tractability by ensuring that the boundaries of $A(y_d)$ and therefore $U(y_d, y_p)$ are smooth on each moderate interval and over centrists, with discontinuous derivatives only at $y_d \in \{y_\ell, y_r\}$. Moreover, $\underline{x}(y_d)$ is strictly concave and $\overline{x}(y_d)$ convex on each interval. We first use this to characterize Y^* and the image of y_d^* more sharply than in the general case and then to study how changes in ρ affect $y_d^*(y_p)$.

Unique Locus of Attraction

The monotonicity of $A(y_d)$ on the interval of centrists ensures that $U(y_d, y_p)$ is strictly quasi-concave on $[y_\ell, y_r]$, so each y_p has a unique local optimum in $[y_\ell, y_r]$. Since $y_d^*(y_d) \subset [y_\ell, y_r]$ for all $y_p \in (\underline{y}_p, \overline{y}_p)$, it follows that $y_d^*(y_p)|_{(\underline{y}_p, \overline{y}_p)}$ is single-valued. Moreover, we can show that $Y^* = \{y^*\}$ is a singleton which all principals in (E_L, E_R) bias their representative towards. Thus, we refer to y^* as the *locus of attraction*.

To characterize y^*, suppose that there there is a $y_p' \in (y_\ell, y_r)$ such that $y_d^*(y_p') = y_p'$. Since U is strictly quasi-concave on $[y_\ell, y_r]$, this implies that

$$0 = \frac{\partial U(y_d, y_p')}{\partial y_d}|_{y_d = y_p'} = \rho_L \frac{\partial u(\underline{x}(y_d), y_p')}{\partial \underline{x}} \frac{\partial \underline{x}(y_d)}{\partial y_d}\Big|_{y_d = y_p'} + \rho_R \frac{\partial u(\overline{x}(y_d), y_p')}{\partial \overline{x}} \frac{\partial \overline{x}(y_d)}{\partial y_d}\Big|_{y_d = y_p'}. \quad (7)$$

At $y_d = y_p'$, the marginal effect of y_d on $U(y_d, y_p')$ through d's proposal is zero. Since both boundaries strictly increase in y_d, (7) requires that the marginal gain from pulling one boundary towards y_p must exactly offset her marginal loss from shifting the other boundary away. Furthermore, since $y_d = y_p'$ is the median, both boundaries of $A(y_p')$ are equidistant from y_p', so (7) reduces to:

$$\rho_L \frac{\partial \underline{x}(y_d)}{\partial y_d}\Big|_{y_d = y_p'} - \rho_R \frac{\partial \overline{x}(y_d)}{\partial y_d}\Big|_{y_d = y_p'} = 0. \quad (8)$$

We define a function that represents the effect in (8) as a function of y_p. Specifically, define $\lambda : [y_l, y_r] \to \mathbb{R}$ as

$$\lambda(y_p) \equiv \rho_L \frac{\partial \underline{x}(y_d)}{\partial y_d}\Big|_{y_d = y_p} - \rho_R \frac{\partial \underline{x}(y_d)}{\partial y_d}\Big|_{y_d = y_p} \quad (9)$$

for $y_p \in (y_\ell, y_r)$, then set $\lambda(y_\ell) = \lim_{y_p \to y_\ell^+} \lambda(y_p)$ and $\lambda(y_r) = \lim_{y_p \to y_r^-} \lambda(y_p)$. This function has two properties that together help us characterize y^*. First, it is strictly decreasing in y_p due to the concavity of $\underline{x}(y_d)$ and convexity of $\overline{x}(y_d)$. Second, its sign indicates which way P wants to bias y_d. For example, $\lambda(y_p) > 0$ implies rightward bias is optimal — since shifting y_d rightward from y_p will make P better off by decreasing the expected distance between y_p and boundary proposals. Proposition 2 uses λ to show that y^* is unique and provides simple conditions to locate it.

Proposition 2. *Suppose Assumptions 1-2. Over $y_p \in (E_L, E_R)$, the optimal representative correspondence y_d^* has a unique fixed point, y^*. Moreover, (i) $\lambda(y_\ell) \leq 0$ implies $y^* = y_\ell$; (ii) $\lambda(y_r) \geq 0$ implies $y_r = y^*$; and (iii) otherwise, $y^* \in (y_\ell, y_r)$.*

On both sides, fringe centrists want to bias inwards — implying $y^* \in (y_\ell, y_r)$ — if the signs of $\lambda(y_\ell)$ and $\lambda(y_r)$ differ. Otherwise, all centrists want to bias in the same direction, so y^* is on the boundary that does not want to bias inward.

Corollary 2. *If $y_p \in (E_L, E_R)$, then the principal's optimal representative is biased strictly towards y^*.*

We know from properties of λ that all centrists bias towards y^* and the monotonicity of y_d^* implies that all moderates do too. Moreover, principals closer to y^* want a representative who is closer to y^*.

Dead zone representatives

Proposition 3 uses λ to characterize (i) whether any moderates on either side want a centrist representative and (ii) which side, if any, has a *dead zone*, denoted Δ, of representatives who are not optimal for any principal.

Proposition 3. *Suppose Assumptions 1-2. In equilibrium,*

1. *$\lambda(y_\ell) \leq 0$ implies $\underline{y}_p = y_\ell < y_r < \overline{y}_p$, so $y_r \in \Delta$ but y_ℓ is not;*
2. *$\lambda(y_r) \geq 0$ implies $\underline{y}_p < y_\ell < y_r = \overline{y}_p$, so $y_\ell \in \Delta$ but y_r is not; and*
3. *otherwise, $\underline{y}_p < y_\ell < y_r < \overline{y}_p$, so $\{y_\ell, y_r\} \subset \Delta$.*

The key factor underlying Proposition 3 is whether centrists at y_ℓ or y_r want to bias inwards. If either does, then its nearby moderates also want to bias inward enough to have a centrist representative. Thus, the sign of $\lambda(y_\ell)$ characterizes whether $\underline{y}_p < y_\ell$ and similarly

$\lambda(y_r)$ characterizes whether $y_r < \overline{y}_p$. If $\lambda(y_r) < 0 < \lambda(y_\ell)$, then moderates near y_ℓ want to bias rightward into (y_ℓ, y_r) and symmetrically for moderates near y_r, so $y^* \in (y_\ell, y_r)$. If not, then one of y_ℓ or y_r wants to self-represent — i.e., $y^* \in \{y_\ell, y_r\}$ — and none of their nearby moderates want a centrist, but some moderates on the other side will want a centrist.

Proposition 3 implies that representatives at y_ℓ and y_r can be optimal for (i) nobody, (ii) exactly one P, or (iii) an interval of centrist P. Notably, they are the only representatives who can be uniquely optimal for more than one P.

Effects of extremism

We have shown an incentive to use strategic representation to counteract extremists. We now show how that varies with changes in relative extremism. Specifically, Proposition 4 characterizes how shifting proposal rights between L and R affects the locus of attraction, y^*, and the set principals who do not choose an extremist, (E_L, E_R). Let $\rho_E \equiv \rho_L + \rho_R$ denote the cumulative proposal rights of the extreme exogenous legislators.

Proposition 4. *Suppose Assumptions 1-2 and assume that ρ_E is fixed so that $\rho_R = 1 - \rho_L$. Increasing ρ_L:*

1. *weakly increases the locus of attraction, y^*; and*
2. *weakly increases the set of principals who strictly prefer a non-extremist, (E_L, E_R), in the strong set order sense.*

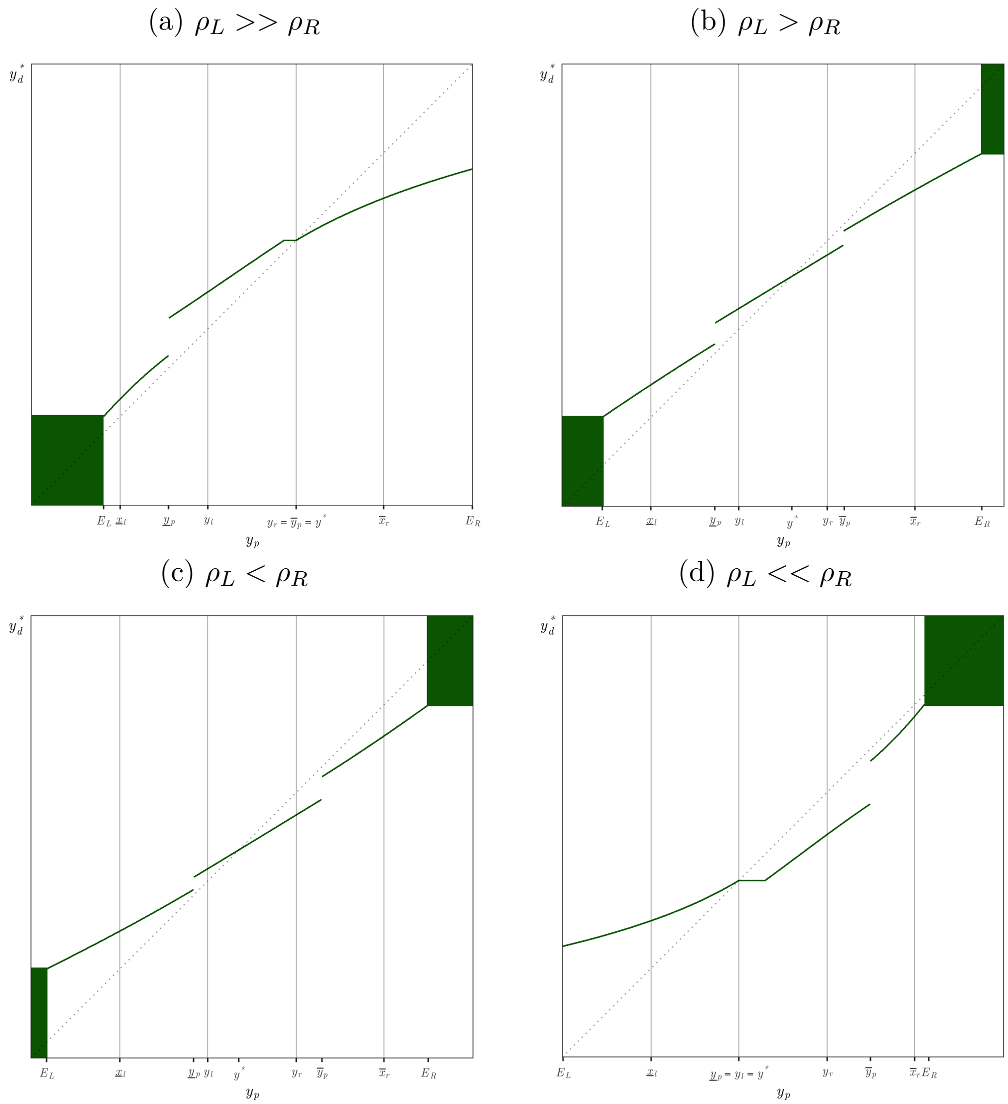
Figure 4 illustrates Proposition 4. Given ρ_E, transferring recognition probability between L and R does not affect the acceptance set on its own, since the median is indifferent between their proposals. Yet, this transfer does affect P's delegation incentives. For example, increasing ρ_L at ρ_R's expense amplifies P's sensitivity to constraining left-extremists but also dampens her sensitivity to constraining right-extremists, and vice versa. Depending on the location of y_p, these effects can change P's optimal representative in different ways.

First, y^* shifts rightward since increasing ρ_L strengthens centrists' desire to constrain left-extremists by skewing rightward.[15] Essentially, centrists want to constrain both extremists but grow more concerned about constraining the strengthened side and less concerned about the weakened side.

Next, E_L and E_R both shift rightward since increasing ρ_L makes right-extremists more inclined to moderate and left-extremists less inclined. Extremists want to constrain their opposing extremist but not their aligned extremist. Thus, their desire to moderate varies with

[15] To see this formally, notice in (9) that $\lambda(y_d)$ increases with $\rho_L = 1 - \rho_R$ for all y_p.

Figure 4: Optimal representatives vary with extremist proposal rights



Note: Figure 4 illustrates Proposition 4 to show how optimal representatives change as recognition probability is transferred between the extremist politicians. Each panel depicts a 5-member legislature where $\delta = .99$, $(y_1, ..., y_4) = (0, .4, .6, 1)$, $\rho_d = 0.1$, and $\rho_\ell = \rho_r = 0.25$. Across panels (a)-(d), $\rho_L = 1 - \rho_R$ varies as follows: (a) .32, (b) .25 (c) .15, and (d) .08.

relative extremism differently if they are on the strengthened side rather than the weakened one. On the strengthened side, moderating is more appealing because constraining their opposing extremists yields a greater return and is also less expensive than constraining their aligned extremists. On the weakened side, moderating is less appealing since these effects reverse.

Fixed Median

Next we consider a special case of Assumption 2 where $y_\ell = y_r$. In this case there is, essentially, a fixed median since $y_m = y_\ell = y_r$ for all y_d. We first show that if, additionally, the proposal rights of the left- and right-extremist legislators are balanced, i.e., $\rho_L = \rho_R$, then y_d^* has a tractable closed-form expression. We then use this to study the value of optimal delegation by comparing a principal's payoff from self-representation to her expected payoff from $y_d^*(y_p)$ and then examining how this benefit of delegation varies with y_p. Finally, we consider an extension of the model in the fixed median case where two ideologically opposed principals each nominate a representative and study their mutually optimal choices.

Optimal Representative Function

We can further sharpen our characterization of y_d^* if the median is fixed ($y_\ell = y_r$) and the proposal rights of extremists is balanced ($\rho_L = \rho_R$). Under these conditions, y_d^* is continuous and over (E_L, E_R) is a convex combination of y_p and y_m, where $\delta\rho_E$ is the weight on y_m.

Corollary 3. *Suppose Assumptions 1-2. If $y_\ell = y_r$ and $\rho_L = \rho_R$, then* $y_d^*(y_p)\Big|_{(E_L,E_R)} = (1-\delta\rho_E)y_p + \delta\rho_E y_m$.

Using Corollary 3, we know $y_m = y^* = \underline{y}_p = \overline{y}_p$. Moreover, we can shed light on how far P moderates, $|y_d^*(y_p) - y_p|$. The effect of ρ_E highlights the key force for moderation: P moderates to constrain extremists and thus moderates further as extremists gain proposal rights. Additionally, more centrist principals do not moderate as far, since biasing y_d towards y_m has a weaker effect on $A(y_d)$. Essentially, the "price" of moderating extremist proposals rises as y_d gets closer to y_m. Finally, increasing δ induces P to moderate further. As the median becomes more patient, there is a drop in the "price" of constraining extremists — increasing δ makes m's expectations about future policymaking more prominent in his voting calculus and thus magnifies the effect of y_d on the acceptance set.

Value of Delegation

Next, we study how much the principal gains from having her optimal representative, relative to having an ally representative, i.e., a representative who shares her ideal point.

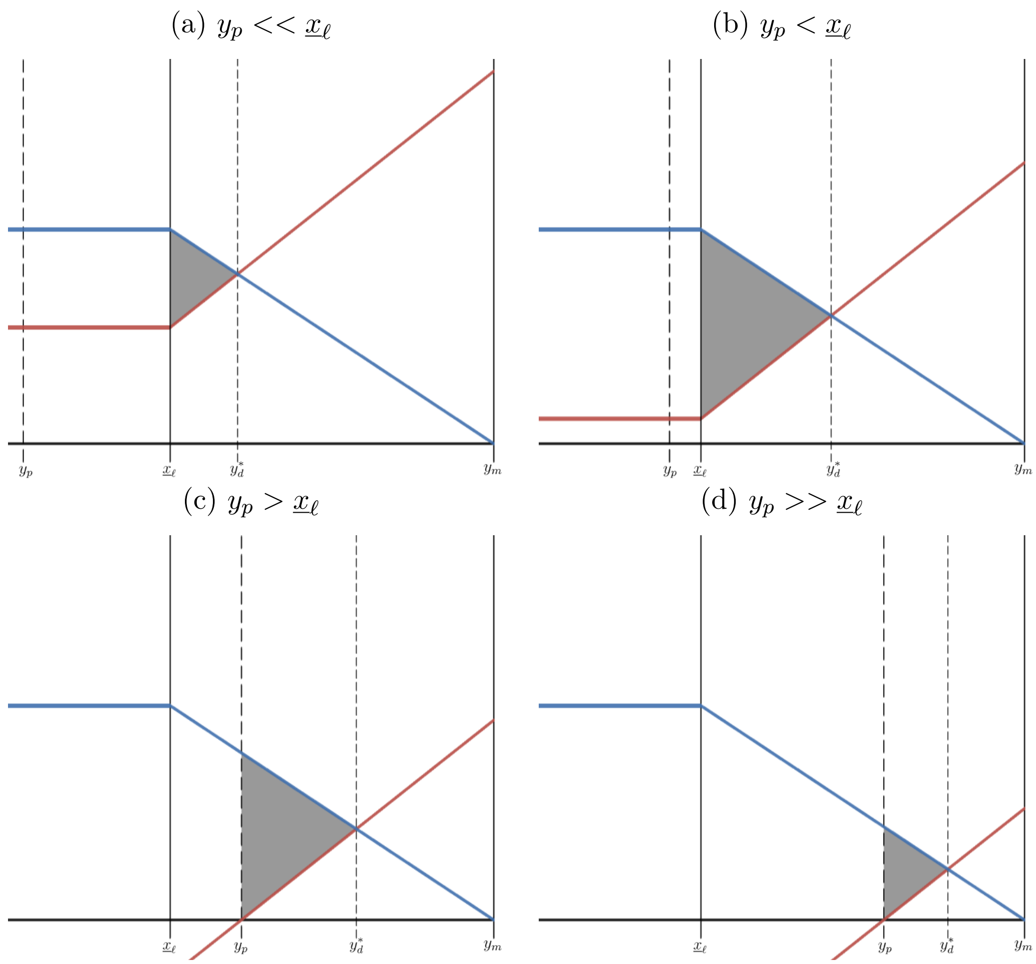
Specifically, we define P's *value of representation* as

$$\nu(y_p) \equiv U(y_d^*(y_p), y_p) - U(y_p, y_p). \tag{10}$$

We focus on the same conditions as in Corollary 3: a fixed median ($y_\ell = y_r$) and balanced extremist proposal rights ($\rho_L = \rho_R$). We show that extremists benefit more from optimal representation as they become more moderate and, conversely, moderates benefit more as they become more extreme. Thus, on each side of the spectrum, the value of representation is highest for principals on the extremist-moderate boundary — i.e., $y_p \in \{\underline{x}_\ell, \overline{x}_r\}$. Figure 5 illustrates.

Proposition 5. *Suppose Assumptions 1-2. If $\rho_\ell = \rho_r$ and $\rho_L = \rho_R$, then ν is strictly increasing on $[E_L, \underline{x}_\ell]$, strictly decreasing on $[\underline{x}_\ell, y_m]$ and analogously for $y_p \in [y_m, E_R]$.*

Figure 5: How the value of representation varies with y_p



Note: Figure 5 displays the value of representation (ν) for four different values of $y_p \in (E_L, y_m)$. In each panel, $\nu(y_p)$ equals the area of the shaded region between the two curves, which are the marginal benefit (downward sloping) and marginal cost (upward sloping) of moderation as functions of y_d.

To characterize ν, we exploit the property that P's optimal representative, $y_d^*(y_p)$, always balances her marginal benefit of moderation against her marginal cost. To illustrate more precisely, if a left-leaning P moderates further from any $y_d \in (\underline{x}_\ell, y_m)$ then she enjoys marginal benefit $\frac{\delta \rho_E \rho_d (y_m - y_d)}{1 - \delta \rho_E}$, which is her gain from further constraining extremist proposals, and incurs marginal cost $\rho_d(y_d - y_p)$, which is her loss from shifting d's proposal further away. Since (i) P's marginal benefit decreases in y_d over this interval and is independent of y_p, while (ii) P's marginal cost increases in y_d and decreases in y_p,[16] the marginal benefit exceeds the marginal cost for $y_d < y_d^*(y_p)$ and vice versa, with the difference increasing in their distance. Thus, $\nu(y_p)$ equals the area between P's marginal cost and benefit curves over $y_d \in [\max\{\underline{x}_\ell, y_p\}, y_d^*(y_p)]$.[17]

If P is sufficiently extreme, $y_p \notin (E_L, E_R)$, she nominates an aligned extremist and thus $\nu(y_p) = 0$. As P becomes less extreme, (i) her marginal cost curve shifts down and (ii) y_d^* shifts towards y_m, which increases the difference between marginal benefit and marginal cost at all $y_d \in [\underline{x}_\ell, y_d^*(y_p)]$, so her value of delegation rises. Figure 5a–5b illustrates.

For moderate P, the acceptance set induced by their ally will shrink as y_p approaches y_m, so there is a smaller difference between P's marginal benefit and marginal cost at every $y_d \geq y_p$. Since the extent of P's optimal bias also decreases, the value of representation decreases as y_p approaches y_m. Figure 5c–5d illustrates.

Competitive Representation

Thus far, we have focused on a principal filling one position and fixed the rest of the political environment. This can reflect situations in which other politicians are already in office, but our analysis also highlights that incentives for moderation will arise in situations where multiple positions will be filled simultaneously (as noted by, e.g., Gailmard and Hammond 2011). In this section, we explore whether those incentives will strengthen or weaken by extending the fixed-median case of the model so that two principals simultaneously pick their representatives.

We extend the model to have two principals, P_a and P_b, each simultaneously appointing representatives, a and b, to fill two positions in a five-player body. The three other politicians are two extremists, L and R, and a veto player, M, who determines whether any proposal passes. Finally, we assume $y_L < y_{p_a} < y_M < y_{p_b} < y_R$, where y_{p_a} and y_{p_b} denote the principals'

[16] As y_p moderates, the marginal benefit from constraining R's proposal decreases relative to constraining L. Since the boundaries of $A(y_d)$ contract at the same rate, if $\rho_L = \rho_R$ then the loss in marginal benefit from shifting $\overline{x}(y_d)$ is exactly equal to the gain from shifting $\underline{x}(y_d)$.

[17] This follows from the fundamental theorem of calculus:

$$\nu(y_p)\big|_{(E_L, \underline{x}_\ell)} = \int_{\underline{x}_\ell}^{y_d^*(y_p)} \left(\frac{\delta \rho_E \rho_d (y_m - y_d)}{1 - \delta \rho_E} - \rho_d (y_d - y_p) \right) \mathrm{d}y_d,$$

and

$$\nu(y_p)\big|_{(\underline{x}_\ell, y_m)} = \int_{y_p}^{y_d^*(y_p)} \left(\frac{\delta \rho_E \rho_d (y_m - y_d)}{1 - \delta \rho_E} - \rho_d (y_d - y_p) \right) \mathrm{d}y_d.$$

ideal points.

By Lemma 1, each $(y_a, y_b) \in X^2$ induces a unique distribution over policy outcomes characterized by the equilibrium acceptance set, $A(y_a, y_b)$. To keep the analysis clean, we assume (i) y_{p_a} and y_{p_b} are both always inside the acceptance set, while (ii) y_L and y_R are always outside.

Our characterization of optimal representatives in the baseline analysis also characterizes best responses in this competitive setting. Since both principals are moderates, each will bias their representative towards M in equilibrium, so $y_{p_a} < y_a^* < y_M < y_b^* < y_{p_b}$. Furthermore, with quadratic policy utility, Proposition 1 implies that each principal always has a unique best response. Specifically, principal P_a's best response to y_b, denoted $y_a(y_b)$, is the unique $y_a \in (y_{p_a}, y_M)$ satisfying the first-order condition:

$$\frac{\partial \underline{x}(y_a, y_b)}{\partial y_a}\Big(\rho_L(y_{p_a} - \underline{x}(y_a, y_b)) + \rho_R(\overline{x}(y_a, y_b) - y_{p_a})\Big) - \rho_a(y_a - y_{p_a}) = 0, \tag{11}$$

and P_b's best response function is analogous.

Lemma 5 establishes that each principal's best response is monotone. Moreover, the direction is determined by which extremist has greater proposal rights.

Lemma 5. *If $\rho_L < \rho_R$, then y_a is strictly decreasing and y_b is strictly increasing; and vice versa if $\rho_L > \rho_R$. If $\rho_L = \rho_R$, then $y_{d_i}(y_{d_{-i}}) = (1 - \delta\rho_E)y_{p_i} + \delta\rho_E y_m$ for all y_{-i}.*

Lemma 5 implies the principals' best responses intersect once. Thus, a unique pair of representatives is mutually optimal and each is strictly more centrist than their principal.

Proposition 6. *There is a unique equilibrium, in which $y_a^* \in (y_{p_a}, y_m)$ and $y_b^* \in (y_m, y_{p_b})$.*

Additionally, Lemma 5 implies that the principal aligned with weaker extremists will moderate further in the competitive setting than in the baseline setting, whereas the principal aligned with stronger extremists will moderate less. Moreover, note that for $\rho_L = \rho_R$, each principal's optimal representative is constant in their opponent's action and coincides with their optimal choice in the single-principal model from Corollary 3.

Corollary 4. *In equilibrium: (i) $\rho_L < \rho_R$ implies $y_a(y_b) < y_a^* < y_M < y_b(y_a) < y_b^*$; (ii) $\rho_L = \rho_R$ implies $y_a(y_b) = y_a^* < y_M < y_b(y_a) = y_b^*$; and (iii) $\rho_L > \rho_R$ implies $y_a^* < y_a(y_b) < y_M < y_b^* < y_b(y_a)$.*

Corollary 4 is driven by the two effects of opponent moderation. To fix ideas, consider shifting y_b inwards. One effect is that extremist proposals also shift inwards, which directly benefits P_a and decreases her marginal benefit from shifting y_a inward. Through this effect, moderation by P_b *substitutes* for moderation by P_a. The other effect is that the acceptance set becomes more sensitive to y_a, i.e., $\frac{\partial^2 \underline{x}(y_a,y_b)}{\partial y_a \partial y_b} < 0$. Through this channel, moderation by P_b reduces the "price" of moderating extremist proposals and *complements* moderation by P_a.

Which effect dominates depends on the balance of extremist proposal rights. The complementary effect dominates on the weak side and conversely on the strong side. If $\rho_L = \rho_R$, then the increased marginal elasticity of A to y_{d_i} exactly offsets the decreased marginal benefit of moderation. As one extremist gains proposal rights at the expense of the other, since each principal's aligned extremist is closer to her ideal point than the non-aligned extremist, the marginal benefit of moderation declines in $y_{d_{-i}}$ at a slower rate for the weak-side principal than the strong-side principal.

Mass Representation

We conclude our analysis by studying the model's implications for collective choice over representatives. So far, we have focused on a single principal choosing a representative. But representatives are often chosen by groups — e.g., voters, parties, etc. We consider an extension of the general baseline model where, in the appointment stage, a group of 3 or more (odd) principals collectively choose the representative. We specifically focus on a setting where there are two exogenous candidates d and d' with arbitrary ideal points y_d and y_d'. The representative is chosen by a single simple-majority vote of the principals. The bargaining stage is identical to the baseline model. All principals are policy-motivated and quadratic loss functions represent their policy utility.

Our objective is to characterize the principals' collective choice of representative. A key property is verifying when collective choice always coincides with the choice of a single, *decisive* principal. Choosing between two representatives is a choice between policy lotteries which are not guaranteed to be ordered with respect to the representative's ideal points. With quadratic policy utility though, preferences over an arbitrary pair of lotteries are *order restricted* — i.e., for any pair of candidates y_d and y_d', the set of y_p for which P prefers y_d and the set of y_p that prefers y_d' are intervals (Duggan 2014; Kartik et al. 2024). Thus the collectively chosen candidate coincides with the candidate preferred by the median principal.

Remark 2. *The median principal is decisive under majority rule.*

But a decisive median does not, on its own, imply whether the set of principals who vote against the median's preferred candidate (if this set is non-empty) is consistently ordered. Proposition 7 establishes that a weak condition on extremist proposal rights ensures that coalitions have a natural ordering. Specifically, any pair of candidate representatives will induce a cutpoint such that all rightward principals prefer the right candidate, and vice versa unless the recognition probability distribution is sufficiently skewed.

Proposition 7. *If* $\max\{\tilde{P}(y_\ell), 1-\tilde{P}(y_\ell)\} \leq \frac{1}{2\delta}$, *then (i) U satisfies the single-crossing condition on X and (ii) sufficiently right-leaning principals prefer the rightmost candidate in any pairwise comparison.*

Underlying Proposition 7 is an important wrinkle that can arise if the principal is an extremist and her aligned extremists are very likely to propose. Then, in some pairwise comparisons between two moderates who are both on the other side of the spectrum, P may prefer the more extreme candidate over the closer candidate. In this scenario, P faces a trade-off. The more aligned moderate candidate makes a more favorable proposal and constrains her opposing extremists more than the alternative but also constrains her aligned extremists more. Viewed from Lemma 4, the closer candidate to her induces a distribution of policy outcomes with a smaller variance but a more distant mean than the candidate further away. Due to risk aversion, the benefit of constraining her opposing extremists outweighs the cost of constraining aligned extremists unless her aligned extremists have sufficiently high recognition probability. In contrast, whenever P is choosing between two centrist candidates, for instance, extremists prefer the nearest candidate since both extreme proposals move in the same direction as y_d within (y_ℓ, y_r). The condition in Proposition 7 is sufficient for $\mu(y_d)$ to increase on X, thus ensuring U satisfies single crossing on X and ruling out cases of the first type.

Conclusion

We study preferences over representatives who participate in collective policymaking. A key force in our analysis is that a representative's ideology affects legislature-wide expectations about policymaking. This force is present in many contexts and we study its consequences for representation. We show how it has important *anticipation effects* by shaping exactly which policies each politician will support, thereby influencing what would pass and what extremists will propose.

We provide a general logic for why moderate representatives can be appealing. We show that (i) all centrist principals want a centrist representative who will be the median (de facto veto) politician and (ii) all moderate principals want a more centrist representative. Even when they are not the median, their closer alignment improves the median's expectation about proposals and thus narrows what can pass, which constrains extremist politicians.

We focused on a collective policymaking environment governed by simple majority rule. Our results generalize to any *strong* voting rule, since there will be a single decisive principal who will effectively determine what can pass (Duggan 2014). To illustrate, our analysis with a fixed median ($y_\ell = y_r$) in Corollary 3 is equivalent to the principal appointing a proposer into a dictatorial rule setting with the dictator already in place. Studying representation in settings where more than one politician will be decisive is a natural direction for future work. For example, under supermajority rules the acceptance set is determined by two (endogenous) veto players rather than a single median. This adds considerable complexity to the analysis since the boundaries of the acceptance set are characterized by a non-smooth implicit curve defined by a system of nonlinear equations. As such, how optimal representatives depend on the legislature's voting rule remains an open question. Our analysis of optimal representatives under simple majority rule is a natural starting point and our results provide a benchmark for future work on this question.

In addition to our main results, which have implications for representation in separation-of-powers systems and congressional committees, our mass representation extension also suggests avenues for new insights. It has potential implications for studying behavior by voters and elites in elections to positions in collective bodies. Several possibilities include (i) voting in elections for collective policymaking positions (Kedar 2005, 2009; Duch et al. 2010), (ii) electoral competition over those offices (Austen-Smith and Banks 1988; Krasa and Polborn 2018), and (iii) its representativeness (Austen-Smith and Banks 1991). We shed new light on how expectations about collective policymaking can affect incentives of party leaders and voters, thereby influencing who gets nominated and their electoral chances. For instance, we show that unless extremist proposal rights are very high, electoral competition in which each party chooses a candidate representative will feature a unique indifferent voter. Thus, (i) each candidate's win probability is easy to characterize, and (ii) both parties will converge toward the median voter's optimal representative. An interesting complication, however, is that the policymaking environment will not only affect how parties evaluate their own candidates but will also shape their view of opposing candidates. Future work in this direction could build on our foundations in order to explore how elite polarization and extremism affect elections.

Appendix

Contents

A Equilibrium Policymaking

We use the following in our proofs to economize on notation.

Definition 3. *$M_L = [\underline{x}_\ell, y_\ell]$, $M_R = [y_r, \overline{x}_r]$, and $M = M_L \cup M_R$.*

A.1 Proof of Lemma 1.

Follows from Propositions 1–2 in Cardona and Ponsati (2011). □

A.2 Proof of Lemma 2.

From Cardona and Ponsati (2011), the equation

$$u(x, y_\ell)(1 - \delta\rho_d) - \delta\left(\sum_{i\in K:|y_i-y_\ell|\geq|y_\ell-x|}\rho_i u(x, y_\ell) + \sum_{i\in K:|y_i-y_\ell|<|y_\ell-x|}\rho_i u(y_i, y_\ell)\right) = 0 \qquad (12)$$

has exactly two solutions: $\underline{x}_\ell \in (0, y_\ell)$ and $\overline{x}_\ell \in (y_\ell, 1)$ where solutions are interior due to Assumption 1.

Similarly, $\underline{x}_r \in (0, y_r)$ and $\overline{x}_r \in (y_r, 1)$ are the only solutions of

$$u(x, y_r)(1 - \delta\rho_d) - \delta\left(\sum_{i\in K:|y_i-y_r|\geq|y_r-x|}\rho_i u(x, y_r) + \sum_{i\in K:|y_i-y_r|<|y_r-x|}\rho_i u(y_i, y_r)\right) = 0. \qquad (13)$$

Part 1 of the proof shows that $y_d \in (\underline{x}_\ell, \overline{x}_r)$ implies $y_d \in \text{int } A(y_d)$. Part 2 shows that $y_d \in \text{int } A(y_d)$ implies $y_d \in (\underline{x}_\ell, \overline{x}_r)$. To show each direction, we use contraposition.

Part 1. Consider $y_d \leq \min A(y_d) = \underline{x}(y_d)$. Then $y_d \leq \underline{x}(y_d) < y_m$, so Assumption 1 implies that $\underline{x}(y_d) \in (0, y_\ell)$ and must solve (12). Thus, $\underline{x}(y_d) = \underline{x}_\ell$. Analogously using (13), $y_d \geq \max A(y_d) = \overline{x}(y_d)$ implies that $\overline{x}(y_d) = \overline{x}_r$. We have shown that $y_d \notin (\underline{x}(y_d), \overline{x}(y_d))$ implies $y_d \notin (\underline{x}_\ell, \overline{x}_r)$. By contraposition, $y_d \in (\underline{x}_\ell, \overline{x}_r)$ implies $y_d \in (\underline{x}(y_d), \overline{x}(y_d)) = \text{int } A(y_d)$.

Part 2. Consider $y_d \leq \underline{x}_\ell$. Then, uniqueness of $A(y_d)$ implies that $y_d \notin \text{int } A(y_d)$ is equivalent to the lower solution of (12) satisfying $\underline{x}_\ell \geq y_d$. Thus, $y_d \leq \underline{x}_\ell$ implies $y_d \leq \min A(y_d)$. An analogous argument shows that $y_d \geq \overline{x}_r$ implies $y_d \geq \max A(y_d)$. We have shown $y_d \notin (\underline{x}_\ell, \overline{x}_r)$ implies $y_d \notin \text{int } A(y_d)$. By contraposition, $y_d \in \text{int } A(y_d)$ implies $y_d \in (\underline{x}_\ell, \overline{x}_r)$. □

A.3 Proof of Lemma 3

Lemma 2 implies Parts 1 and 5. We first prove Parts 2–4. We then prove that $\underline{x}_\ell \leq \underline{x}_r$ and $\overline{x}_\ell \leq \overline{x}_r$, which combines with Parts 1–5 to directly imply $A(y_d) \subset [\underline{x}_\ell, \overline{x}_r]$.

To prove Parts 2–4, recall that Assumption 1 implies $A(y_d) \subset (0,1)$ for all y_d. With quadratic loss, this implies that for each $y_d \in X$ there is a ϕ such that $A(y_d) = [y_m - \phi, y_m + \phi]$ where $0 < \phi < \min\{y_m, 1 - y_m\}$ and $u(y_m, y_m \pm \phi) = \delta V_m(y_m - \phi, y_m + \phi)$. Note that $u(y_m, y_m \pm \phi) = u(0, \phi) = 1 - \phi^2$. Let

$$U_m(\phi) = \sum_{i \in N : |y_m - y_i| \leq \phi} \rho_i u(y_i, y_m) + \sum_{i \in N : |y_m - y_i| > \phi} \rho_i (1 - \phi^2).$$

Note that for an arbitrary y_d, $U_m(0) = \delta > 0$ and $U_m(\phi)$ is continuously decreasing and differentiable a.e. in ϕ.[18] Define

$$\phi(y_d) = \phi \in (0, \min\{y_m, 1 - y_m\}) \text{ such that } 1 - \phi^2 = U_m(\phi).$$

Since $\underline{x}(y_d) = y_m - \phi(y_d)$ and $\overline{x}(y_d) = y_m + \phi(y_d)$, it is sufficient to show that $\phi(y_d)$ is strictly decreasing in $|y_d - y_m|$ at a rate that approaches zero as $y_d \to y_m$ on M to prove Parts 2 and 4 of Lemma 3. Direct computations show that a.e. on M,

$$\phi'(y_d) = \frac{\delta \rho_d (y_d - y_m)}{\left(1 - \delta \sum_{i \in N : |y_m - y_i| > \phi} \rho_i\right)}.$$

Note that $\phi'(y_d) < 0$ if $y_d < y_m$ and $\phi'(y_d) > 0$ if $y_m < y_d$. Since $m = \ell$ if $y_d < y_m$ and $m = r$ if $y_d > y_m$, we have that $\phi(y_d)$ is strictly decreasing in $|y_d - y_m|$ on M. Moreover, $\lim_{y_d \to^- y_\ell} \phi'(y_d) = 0$ and $\lim_{y_d \to^+ y_r} \phi'(y_d) = 0$.

To prove part 3, note that if $y_d \in (y_\ell, y_r)$, then $d = m$. Direct computations show that a.e. on (y_ℓ, y_r),

$$\phi'(y_d) = \frac{\delta \sum_{i \in N : |y_m - y_i| < \phi} \rho_i (y_d - y_i)}{\left(1 - \delta \sum_{i \in N : |y_m - y_i| > \phi} \rho_i\right)}.$$

Notice that

$$|\phi'(y_d)| < \frac{\delta \phi(y_d) \left(\sum_{i \in N : |y_m - y_i| < \phi} \rho_i \right)}{\left(1 - \delta \sum_{i \in N : |y_m - y_i| > \phi} \rho_i\right)} = \frac{\delta \phi(y_d) \left(1 - \sum_{i \in N : |y_m - y_i| > \phi} \rho_i \right)}{\left(1 - \delta \sum_{i \in N : |y_m - y_i| > \phi} \rho_i\right)}.$$

[18] Left and right derivatives of $U_m(\phi)$ with respect to y_d are unequal only at ϕ such that a $|y_m - y_i| = \phi$ for some $i \in N$.

Because $\phi(y_d) < 1$,

$$\delta < 1 \implies \delta \left(1 - \sum_{i \in N: |y_m - y_i| > \phi} \rho_i\right) < \left(1 - \delta \sum_{i \in N: |y_m - y_i| > \phi} \rho_i\right)$$

implies $|\phi'(y_d)| \in (0, 1)$. Thus $\underline{x}'(y_d) = 1 - \phi'(y_d) > 0$ and $\overline{x}'(y_d) = 1 + \phi'(y_d) > 0$.

Next we prove that $\underline{x}_\ell \leq \underline{x}_r$ and $\overline{x}_\ell \leq \overline{x}_r$. Let

$$U_y(\zeta) = \sum_{i \in K: |y - y_i| \leq \zeta} \rho_i u(y_i, y) + \left(\rho_d + \sum_{i \in K: |y - y_i| > \zeta} \rho_i\right)(1 - \zeta^2)$$

and

$$\zeta(y) = \zeta \in (0, \min\{y, 1-y\}) \text{ such that } 1 - \zeta^2 = U_y(\zeta).$$

Note that $\underline{x}_\ell = y_\ell - \phi(0) = y_\ell - \zeta(y_\ell)$, $\overline{x}_\ell = y_\ell + \phi(0) = y_\ell + \zeta(y_\ell)$, $\underline{x}_r = y_r - \phi(1) = y_r - \zeta(y_r)$, and $\overline{x}_r = y_r + \zeta(y_r)$. It is straightforward to check that $U_y(\zeta)$ is continuous and a.e. differentiable in $y \in [y_\ell, y_r]$ and that $\zeta(y)$ is unique for all $y \in [y_\ell, y_r]$. It is therefore sufficient to prove the result to show that $y - \zeta(y)$ and $y + \zeta(y)$ are increasing on $[y_\ell, y_r]$. Direct computations show that wherever $\zeta(y)$ is differentiable,

$$\zeta'(y) = \frac{\delta \sum_{i \in K: |y - y_i| < \zeta} \rho_i (y - y_i)}{1 - \delta \left(\rho_d + \sum_{i \in K |y - y_i| > \zeta} \rho_i\right)},$$

so

$$|\zeta'(y)| < \frac{\delta \zeta(y) \left(\sum_{i \in K: |y - y_i| < \zeta} \rho_i (y - y_i)\right)}{1 - \delta \left(\rho_d + \sum_{i \in K |y - y_i| > \zeta} \rho_i\right)} \in (0, 1)$$

implies that $y - \zeta(y)$ and $y + \zeta(y)$ are increasing. □

A.4 Proof of Lemma 4

Part 1. We first prove that $[\underline{\pi}, \overline{\pi}]$ exists. On $[y_\ell, y_r]$, both boundaries of $A(y_d)$ and y_d are strictly increasing, so $\mu(y_d)$ is strictly increasing on $[y_\ell, y_r]$. From Lemma 3, $\underline{x}'(y_d) = -\overline{x}'(y_d) \to 0$ as $y_d \to^- y_\ell$ and $y_d \to^+ y_\ell$. This implies that $\mu(y_d)$ is strictly increasing on intervals $(y_\ell - \epsilon_\ell, y_\ell]$ and $[y_r, y_r + \epsilon_r)$ for $\epsilon_\ell > 0$ and $\epsilon_r > 0$. Since $A(y_d)$ is continuous, it follows that $\mu(y_d)$ is strictly increasing on $[y_\ell - \epsilon_\ell, y_r + \epsilon_r]$. Thus there are maximal $\epsilon'_\ell \in (0, y_\ell - \underline{x}_\ell]$ and $\epsilon'_r \in (0, \overline{x}_r - y_r]$ such that $\mu(y_d)$ is strictly increasing on $[y_\ell - \epsilon_\ell, y_r + \epsilon_r]$. Setting $\underline{\pi} = y_\ell - \epsilon'_\ell$ and $\overline{\pi} = y_r + \epsilon'_r$

completes the proof of Part 1.

Part 2. To prove Part 2, we show that $1-\tilde{P}(y_\ell) > \frac{1}{2\delta}$ is a necessary condition for $\underline{\pi} > \underline{x}_\ell$. An analogous argument establishes that $\overline{\pi} < \overline{x}_r$ only if $\tilde{P}(y_\ell) > \frac{1}{2\delta}$. Since these two conditions cannot be simultaneously satisfied, it follows that $\underline{\pi} > \underline{x}_\ell$ implies $\overline{\pi} = \overline{x}_r$ and that $\overline{\pi} < \overline{x}_r$ implies $\underline{\pi} = \underline{x}_\ell$.

To show that $\underline{\pi} > \underline{x}_\ell$ implies $1-\tilde{P}(y_\ell) > \frac{1}{2\delta}$, suppose that $\underline{\pi} > \underline{x}_\ell$. By construction, $\mu(y_d)$ is non-monotonic on $[\underline{x}_\ell, \underline{\pi}]$. Since $\mu(y_d)$ is continuous and differentiable almost everywhere, there must exist an open interval $Z \subset [\underline{x}_\ell, \underline{\pi}]$ on which

$$\begin{aligned}
\mu'(y_d) &= \underline{x}'(y_d)P(\underline{x}(y_d)) + \overline{x}'(y_d)[1-P(\overline{x}(y_d))] + \rho_d \\
&= \delta\rho_d(y_\ell - y_d)\left(\frac{P(\underline{x}(y_d)) - [1-P(\overline{x}(y_d))]}{1-\delta[P(\underline{x}(y_d)) + 1 - P(\overline{x}(y_d))]}\right) + \rho_d < 0 \\
\implies\ 1 &< \delta(y_\ell - y_d)\left(\frac{[1-P(\overline{x}(y_d))] - P(\underline{x}(y_d))}{1-\delta[P(\underline{x}(y_d)) + 1 - P(\overline{x}(y_d))]}\right).
\end{aligned}$$

Since $0 < y_\ell - y_d < 1$, it must be that $P(\underline{x}(y_d)) < [1-P(\overline{x}(y_d))]$ and

$$1 < \delta\left(\frac{[1-P(\overline{x}(y_d))] - P(\underline{x}(y_d))}{1-\delta[P(\underline{x}(y_d)) + 1 - P(\overline{x}(y_d))]}\right),$$

which implies

$$\frac{1}{2\delta} < [1-P(\overline{x}(y_d))].$$

Because $y_\ell < \overline{x}(y_d)$, the last inequality can be satisfied for some $y_d \in M_L$ only if $\frac{1}{2\delta} < 1-\tilde{P}(y_\ell)$.

Part 3. We prove the result for an arbitrary interval $Z \subset M_L$ on which $\mu(y_d)$ is decreasing. The proof for $Z \subset M_R$ is analogous. We first show that $y_\ell < \mu(y_d)$. We know from Part 2 of Lemma 4 that $1-P(\overline{x}(y_d)) > \frac{1}{2\delta} > \frac{1}{2}$ at any $y_d \in M_L$ such that $\mu(y_d)$ is non-increasing. Thus for $y_d \in Z$,

$$\begin{aligned}
\mu(y_d) &= \underline{x}(y_d)P(\underline{x}(y_d)) + \overline{x}(y_d)[1-P(\overline{x}(y_d))] + \sum_{i\in N: y_i \in (\underline{x}(y_d), \overline{x}(y_d)]} \rho_i y_i \\
&> \underline{x}(y_d)[1-(1-P(\overline{x}(y_d)))] + \overline{x}(y_d)[1-P(\overline{x}(y_d))] \\
&= [y_\ell - \phi(y_d)]P(\overline{x}(y_d)) + [y_\ell + \phi(y_d)][1-P(\overline{x}(y_d))] \\
&= y_\ell + \phi(y_d)[1-2P(\overline{x}(y_d))] \\
&> y_\ell.
\end{aligned}$$

Next we show that $\sigma^2(y_d)$ is strictly decreasing in y_d on Z. Because $\mu(y_d)$ is continuous and a.e. differentiable, $\sigma^2(y_d)$ must be too. Since $y_d \in M_L$,

$$u(\overline{x}(y_d), y_\ell) = \delta V_\ell(\underline{x}(y_d), \overline{x}(y_d))$$

$$\implies 1 - (\overline{x}(y_d) - y_\ell)^2 = \delta[1 - (\mu(y_d) - y_\ell)^2 - \sigma^2(y_d)]$$

$$\implies \delta\sigma^2(y_d) = \delta[1 - (y_\ell - \mu(y_d))^2] - [1 - (\overline{x}(y_d) - y_\ell)^2].$$

Thus almost everywhere on Z,

$$\frac{\partial \delta\sigma^2(y_d)}{\partial y_d} = -2\delta[\mu(y_d) - y_\ell]\mu'(y_d) + 2[\overline{x}(y_d) - y_\ell]\overline{x}'(y_d) < 0$$

if

$$\overline{x}'(y_d)[\overline{x}(y_d) - y_\ell] - \delta\mu'(y_d)[\mu(y_d) - y_\ell] < 0. \tag{14}$$

We already showed $\mu'(y_d) \leq 0$ implies $y_\ell < \mu(y_d) < \overline{x}(y_d)$. Therefore

$$\overline{x}'(y_d) - \delta\mu'(y_d) < 0 \tag{15}$$

implies (14). Substituting

$$\mu'(y_d) = \overline{x}'(y_d)[1 - P(\overline{x}(y_d))] + \underline{x}'(y_d)P(\underline{x}(y_d)) + \rho_d = \overline{x}'(y_d)[1 - P(\overline{x}(y_d)) - P(\underline{x}(y_d))] + \rho_d$$

into (15) yields

$$\overline{x}'(y_d) - \delta\mu'(y_d) = \overline{x}'(y_d)[1 - \delta(1 - P(\overline{x}(y_d)) - P(\underline{x}(y_d)))] - \delta\rho_d < 0.$$

Because $\overline{x}'(y_d) < 0$ by Lemma 3 and $[1 - P(\overline{x}(y_d)) - P(\underline{x}(y_d))] \in (0, 1)$, its follows that (15) is satisfied. Thus $\sigma^2(y_d)$ is almost everywhere strictly decreasing on Z. The continuity of $\sigma^2(y_d)$ then implies that $\sigma^2(y_d)$ is strictly decreasing on Z. □

B Optimal Representatives

We establish general properties of $y_d^*(y_p)$ in Lemma A1. We use Lemmas A2-A4 to prove that $y_d^*(y_d)$ is ordered on X and single-valued almost everywhere on (E_L, E_R).

B.1 General Properties of y_d^*

Lemma A1.

1. *$[0, \underline{x}_\ell] \cap y_d^*(y_p) \neq \emptyset \iff [0, \underline{x}_\ell] \subseteq y_d^*(y_p)$ and $[\overline{x}_r, 1] \cap y_d^*(y_p) \neq \emptyset \iff [\overline{x}_r, 1] \subseteq y_d^*(y_p)$;*

2. $y_p \geq y_\ell$ *implies* $y_d^*(y_p) \cap [\underline{x}_\ell, y_\ell) = \emptyset$, *and* $y_p \leq y_r$ *implies* $y_d^*(y_p) \cap (y_r, \overline{x}_r] = \emptyset$;

3. $y_p < y_\ell$ *implies* $y_\ell \notin y_d^*(y_p)$, *and* $y_p > y_r$ *implies* $y_r \notin y_d^*(y_p)$;

4. $y_p \in (\underline{x}_\ell, y_\ell)$ *implies* $[\underline{x}_\ell, y_p] \cap y_d^*(y_p) = \emptyset$, *and* $y_p \in (y_r, \overline{x}_r)$ *implies* $[y_p, \overline{x}_r] \cap y_d^*(y_p) = \emptyset$; *and*

5. $y_p \in [0, \underline{x}(y_\ell)] \cup [\overline{x}(y_r), 1]$ *implies* $[y_\ell, y_r] \cap y_d^*(y_p) = \emptyset$.

6. *There exist* $e_L < \underline{x}_\ell$ *and* $e_R > \overline{x}_r$ *such that* $y_d^*(y_p) \subset (\underline{x}_\ell, \overline{x}_r)$ *if* $y_d \in (e_L, e_R)$.

Proof.
The principal solves $\max_{y_d \in [0,1]} U(y_d, y_p)$ where $U : [0,1]^2 \to \mathbb{R}$ is:

$$U(y_d, y_p) \equiv P(\underline{x}(y_d))u(\underline{x}(y_d), y_p) + [1 - P(\overline{x}(y_d))]u(\overline{x}(y_d), y_p) + \sum_{i \in N: y_i \in (\underline{x}(y_d), \overline{x}(y_d)]} \rho_i u(y_i, y_p)$$
$$= 1 - (y_p - \mu(y_d))^2 - \sigma^2(y_d).$$

For each part 1–5, we prove one side since the other side is analogous.

1. For $y_d \leq \underline{x}_\ell$, Lemma 3 implies $A(y_d) = [\underline{x}_\ell, \overline{x}_\ell]$, so $U(\underline{x}_\ell; y_p) = U(y_d; y_p)$.

2. Consider $y_p \leq y_r$ and suppose there exists $y_d' \in (y_r, \overline{x}_r]$ such that $y_d' \in y_d^*(y_p)$. We establish a contradiction by showing there must exist $y_d'' < y_r$ such that $U(y''; y_p) > U(y'; y_p)$. First, Lemma 2 implies $y_p < y_d' < \overline{x}(y_d')$.

   Because $\underline{x}(y_d)$ is strictly decreasing and $\overline{x}(y_d)$ strictly increasing on $[y_r, \overline{x}_r]$, we know $y_d' \in y_d^*(y_p)$ requires $y_p \leq \underline{x}(y_d')$ (otherwise, $U(y_d - \varepsilon; y_p) > U(y_d; y_p)$ for some $\varepsilon > 0$). Next, by Lemma 3, we know that (i) $\underline{x}(y_d)$ is continuous and strictly increasing on $[\underline{x}_\ell, y_r]$, (ii) $\underline{x}_\ell < \underline{x}_r$, and (iii) $y_d \leq \underline{x}(y_d)$ if and only if $y_d \leq \underline{x}_\ell$. Thus, a $y_d'' \in (y_p, y_r)$ exists such that $\underline{x}(y_d'') = \underline{x}(y_d')$. Since $\overline{x}(y_d) = 2y_m - \underline{x}(y_d)$ and $y_p < y_d'' < y_r < y_d'$, it follows that $y_p < \overline{x}(y_d'') < 2y_r - \underline{x}(y') = \overline{x}(y_d')$. We have shown that $|y_p - y_d''| < |y_p - y_d'|$, $|y_p - \overline{x}(y_d'')| < |y_p - \overline{x}(y_d')|$, and $|y_p - \underline{x}(y_d'')| = |y_p - \underline{x}(y_d')|$, which implies $U(y_d', y_p) < U(y_d'', y_p)$. But then $y_p' \notin y_d^*(y_p)$, a contradiction.

3. Lemma 3 implies that for $y_p < y_\ell$, $U(y_d; y_p)$ is continuously differentiable on $(y_\ell - \varepsilon, y_\ell)$ with
$$\lim_{y_d \to y_\ell^-} \frac{\partial U(y_d; y_p)}{\partial y_d} = \rho_d \cdot \frac{\partial u(y_d, y_p)}{y_d}\bigg|_{y_d = y_\ell^-} < 0.$$
   Then, continuity of $U(y_d; y_p)$ implies $y_\ell \notin y_d^*(y_p)$.

4. Consider $y_p \in (\underline{x}_\ell, y_\ell)$. Then, $y_p \in \text{int}A(y_d)$ for all $y_d \leq y_p$. By Lemma 3, $\underline{x}(y_d)$ is continuous and strictly increasing on $[\underline{x}_\ell, y_\ell]$, while $\overline{x}(y_d)$ is continuous and strictly decreasing. Thus, there exists $\varepsilon > 0$ such that $P(\underline{x}(y_d))\, u(\underline{x}(y_d), y_p) + [1 - P(\overline{x}(y_d))]\, u(\overline{x}(y_d), y_p)$ is strictly increasing on $[\underline{x}_\ell, y_p + \varepsilon]$. By assumption, $\frac{\partial u(y_d;y_p)}{\partial y_d} > 0$ if $y_d < y_p$ and $\frac{\partial u(y_d;y_p)}{\partial y_d}\Big|_{y_d=y_p} = 0$. By continuity of $\frac{\partial u(y_d;y_p)}{\partial y_d}$ there exists $y_d' \in (y_p, y_p + \varepsilon)$ such that $U(y_d; y_p)$ is strictly increasing over $y_d \in [\underline{x}_\ell, y_d')$.

5. Lemma 3 establishes that $\underline{x}(y_d)$ and $\overline{x}(y_d)$ are strictly increasing on $[y_\ell, y_r]$. Therefore $U(y_d; y_p)$ is strictly decreasing on $[y_\ell, y_r]$ if $y_p \leq \underline{x}(y_\ell)$. Thus $\text{argmax}_{y_d \in [y_\ell, y_r]} U(y_d; y_p) = y_\ell$ for all $y_p \leq \underline{x}(y_r)$. But by part 3, $y_\ell \in y_d^*(y_p)$ only if $y_p \geq y_\ell$.

6. Parts 1–5 imply that $y_d^*(y_p) \subset (\underline{x}_\ell, \overline{x}_r)$ for all $y_p \in [\underline{x}_\ell, \overline{x}_r]$. Part 6 then follows from upper hemicontinuity of y_d^*. □

B.2 Ordering of y_d^*

Lemma A2. *Let $Y \subseteq X$ denote an arbitrary subset of X and define $\tilde{y}_d : X \times 2^X \to 2^X$ as*

$$\tilde{y}_d(y_p; Y) \equiv \operatorname*{argmax}_{y_d \in Y} U(y_d, y_p).$$

If $\mu(y_d)$ is weakly increasing on Y, then $\tilde{y}_d(y_p; Y)$ is increasing in the strong set order sense. If $\mu(y_d)$ is strictly increasing on Y, then every selection from $\tilde{y}_d(y_p; Y)$ is weakly increasing.

Proof. Consider an arbitrary subset $Y \subseteq X$ and arbitrary $y_d, y_d' \in Y$. Since

$$U(y_d', y_p) - U(y_d, y_p) = 1 - (\mu(y_d') - y_p)^2 - \sigma^2(y_d') - [1 - (\mu(y_d) - y_p)^2 - \sigma^2(y_d))]$$
$$\implies \frac{\partial [U(y_d', y_p) - U(y_d, y_p)]}{\partial y_p} \propto \mu(y_d') - \mu(y_d),$$

we have the following:

1. If $y_d < y_d' \implies \mu(y_d) \leq \mu(y_d')$, then $U(y_d, y_p)$ satisfies the single-crossing property.

2. If $y_d < y_d' \implies \mu(y_d) < \mu(y_d')$, then $U(y_d, y_p)$ satisfies the strict single-crossing property.

The result then follows directly from Theorem 4 in Milgrom and Shannon (1994). □

Lemma A3. *$\mu(y_d)$ is strictly increasing on $\mathcal{M}_L \cup [y_\ell, y_r] \cup \mathcal{M}_R$ and weakly increasing on D where*

$$\mathcal{M}_L \equiv \{y_d \in M_L : \mu(y_d) < \mu(y_d') \text{ for all } y_d' \in M_L \text{ such that } y_d < y_d'\},$$
$$\mathcal{M}_R \equiv \{y_d \in M_R : \mu(y_d') < \mu(y_d) \text{ for all } y_d' \in M_R \text{ such that } y_d' < y_d\},$$

and

$$D \equiv \begin{cases} \mathcal{M}_L \cup [y_\ell, y_r] \cup \mathcal{M}_R \cup [\overline{x}_r, 1] & \text{if} \quad \underline{x}_\ell \notin \mathcal{M}_L \\ [0, \underline{x}_\ell] \cup \mathcal{M}_L \cup [y_\ell, y_r] \cup \mathcal{M}_R & \text{if} \quad \overline{x}_r \notin \mathcal{M}_R \\ [0, \underline{x}_\ell] \cup \mathcal{M}_L \cup [y_\ell, y_r] \cup \mathcal{M}_R \cup [\overline{x}_r, 1] & \text{otherwise} \end{cases}$$

Proof. We first show that $\mu(y_d)$ strictly increases on $\mathcal{M}_L \cup [y_\ell, y_r] \cup \mathcal{M}_R$. Both boundaries of the acceptance set strictly increase on $[y_\ell, y_r]$ which implies that $\mu(y_d)$ is strictly increasing on $[y_\ell, y_r]$. Lemma 4 implies that $y_\ell \in \mathcal{M}_L$ and $y_r \in \mathcal{M}_R$. Since $\mu(y_d)$ is strictly increasing on $\mathcal{M}_L$ and $\mathcal{M}_R$, it follows that $\mu(y_d)$ strictly increases on $\mathcal{M}_L \cup [y_\ell, y_r] \cup \mathcal{M}_R$.

We now show that $\mu(y_d)$ is weakly increasing on D. Since $A(y_d)$ is constant on $[0, \underline{x}_\ell]$ and $[\overline{x}_r, 1]$, so is $\mu(y_d)$. Thus if $\underline{x}_\ell \in \mathcal{M}_L$ and $\overline{x}_r \in \mathcal{M}_R$, then (i) $\mu(y_d) = \mu(\underline{x}_\ell) < \mu(y_d')$ for all $y_d \leq \underline{x}_r$ and $y_d' \in D$ such that $y_d' > \underline{x}_\ell$ and (ii) $\mu(y_d') < \mu(\overline{x}_r) = \mu(y_d)$ for all $y_d \geq \overline{x}_r$ and $y_d' \in D$ such that $y_d' < \overline{x}_r$. If $\underline{x}_\ell \notin \mathcal{M}_L$, then $\overline{x}_r \in \mathcal{M}_R$ by Lemma 4 so $\mu(y_d)$ is strictly increasing on $D \setminus (\overline{x}_r, 1]$ and $\mu(y_d) = \mu(\overline{x}_r)$ for all $y_d > \overline{x}_r$. Analogously, if $\overline{x}_r \notin \mathcal{M}_R$, then $\underline{x}_\ell \in \mathcal{M}_L$ so $\mu(y_d)$ strictly increases on $D \setminus [0, \underline{x}_\ell)$ and $\mu(y_d) = \mu(\underline{x}_r)$ for all $y_d < \underline{x}_\ell$. □

Lemma A4. *The image of y_d^* is a subset of D*

Proof. Let $\tilde{Y} = \{y_d^*(y_p) | y_p \in X\}$ denote the image of y_d^* and define $\mathcal{M} \equiv \mathcal{M}_L \cup \mathcal{M}_R$. Lemma A1 implies $\underline{x}_\ell \in \tilde{Y} \iff [0, \underline{x}_\ell] \subset \tilde{Y}$ and $\overline{x}_r \in \tilde{Y} \iff [\overline{x}_r, 1] \subset \tilde{Y}$. By definition, $\underline{x}_\ell \in \mathcal{M} \iff [0, \underline{x}_\ell] \subset D$ and $\overline{x}_r \in \mathcal{M} \iff [\overline{x}_r, 1] \subset D$. We can therefore prove Lemma A4 by showing that $\tilde{Y} \cap \{M \setminus \mathcal{M}\} = \emptyset$.

By definition, $y_d \in \{M \setminus \mathcal{M}\}$ if and only if one of the following holds:

1. If $y_d \in M_L$, then a $y_d' \in M_L$ exists such that $y_d < y_d'$ and either
   (a) $\mu(y_d') = \mu(y_d)$, or
   (b) $\mu(y_d') < \mu(y_d)$.
2. If $y_d \in M_R$, then a $y_d'' \in M_R$ exists such that $y_d'' < y_d$ and either
   (a) $\mu(y_d) = \mu(y_d'')$, or
   (b) $\mu(y_d) < \mu(y_d'')$.

We show $y_d \notin \tilde{Y}$ for all of these cases. We proceed in two steps.

Step 1. We first show that $y_d \notin \tilde{Y}$ if y_d satisfies condition 1(a) or 2(a). To do so, we prove the following:

Suppose $y_d < y_d'$ and $\mu(y_d) = \mu(y_d')$. Then for all y_p, $U(y_d, y_p) < U(y_d', y_p)$ if $y_d, y_d' \in M_L$ and $U(y_d', y_p) < U(y_d, y_p)$ if $y_d, y_d' \in M_R$.

Proof: We prove the claim for $y_d', y_d \in M_L$. An analogous argument establishes the result for $y_d', y_d \in M_R$. For this case, we know $m = \ell$'s continuation value, $V_\ell = 1 - (\mu(y_d) - y_\ell)^2 - \sigma^2(y_d)$, is strictly increasing in y_d on M_L since $\underline{x}(y_d)$ is strictly increasing and $\overline{x}(y_d)$ strictly decreasing. Thus $\mu(y_d) = \mu(y_d')$ implies $\sigma^2(y_d') < \sigma^2(y_d)$. Therefore $U(y_d', y_p) - U(y_d, y_p) = \sigma^2(y_d) - \sigma^2(y_d') > 0$ for all y_p. ■

Step 2. We now show that $y_d \notin y_d^*(X)$ if y_d satisfies condition 1(b) or 2(b). Part 1 of Lemma 4 implies that $M \setminus \mathcal{M} \subseteq [\underline{x}_\ell, y_\ell) \cup (y_r, \overline{x}_r]$. Lemma A1 establishes that $y_d^*(y_p) \cap [\underline{x}_\ell, y_\ell) = \emptyset$ if $y_p \geq y_\ell$ and $y_d^*(y_p) \cap (y_r, \overline{x}_r] = \emptyset$ if $y_p \leq y_r$. It is therefore sufficient to show that $y_d \notin y_d^*(y_p)$ for any $y_p < y_\ell$ if y_d satisfies 1(b) and that $y_d \notin y_d^*(y_p)$ for any $y_p > y_r$ if y_d satisfies 2(b). To show this, we prove the following:

Consider an arbitrary interval $Z = [\underline{z}, \overline{z}] \subset M$ on which $\mu(y_d)$ is decreasing.

1. *If $Z \subset M_L$, then $\operatorname{argmax}_{y_d \in Z} U(y_d, y_p) = \overline{z}$ for all $y_p < y_\ell$.*
2. *If $Z \subset M_R$, then $\operatorname{argmax}_{y_d \in Z} U(y_d, y_p) = \underline{z}$ for all $y_p > y_r$.*

Proof: We prove the first part; the second part is analogous. Consider an arbitrary interval $Z = [\underline{z}, \overline{z}] \subseteq M_L$ such that $\mu(y_d)$ is non-increasing on Z. Since $\mu(y_d)$ is strictly increasing on $[\underline{\pi}, \overline{\pi}]$, it must be that $Z \subset [\underline{x}_\ell, \underline{\pi}]$. Part 3 of Lemma 4 therefore implies that for all $y_d \in Z$, (i) $y_\ell < \mu(y_d)$ and (ii) $\sigma^2(y_d)$ is strictly decreasing. Thus $U(y_d, y_p) = 1 - (\mu(y_d) - y_p)^2 - \sigma^2(y_d)$ is strictly increasing on Z for all $y_p < y_\ell$. ■ □

Lemma A5. *The correspondence y_d^* is increasing in $y_p \in X$ in the strong set order sense and $y_d^*|_{(E_L, E_R)}$ is a singleton almost everywhere.*

Proof. Lemma A4 implies that $y_d^*(y_p) = \tilde{y}_d(y_p; D)$. Since $\mu(y_d)$ is increasing on D by Remark A3, Lemma A2 implies that $\tilde{y}_d(y_p; D)$ is increasing in the strong set order sense, so $y_d^*(y_p)$ is too. Lemmas A1 and A4 imply $y_d^*|_{(E_L,E_R)}(y_p) = \tilde{y}_d|_{(E_L,E_R)}(y_p; \mathcal{M} \cup [y_\ell, y_r])$. Since $\mu(y_d)$ is strictly increasing on $\mathcal{M} \cup [y_\ell, y_r]$ by Remark A3, Lemma A2 implies that every selection from $\tilde{y}_d|_{(E_L,E_R)}(y_p; \mathcal{M} \cup [y_\ell, y_r])$ is increasing and therefore the same must be true of $y_d^*|_{(E_L,E_R)}(y_p)$. Since y_d^* is upper hemicontinuous and increasing in the strong set order sense, it follows that $y_d^*|_{(E_L,E_R)}(y_p)$ is a singleton almost everywhere (Kenderov 1976). □

B.3 Proof of Proposition 1

Follows immediately from Lemmas A1–A5. □

C Polarized Legislature

First, in Lemma A6 we refine our characterization of $\underline{x}(y_d)$ and $\overline{x}(y_d)$ under Assumptions 1 and 2. In Lemmas A7 and A8, we characterize the principal's locally optimal representative within the set of centrist representatives and aligned moderate representatives, respectively. We use these to prove Propositions 2–4.

Lemma A6. *Suppose Assumptions 1 and 2. Then:*

$$\underline{x}_\ell = y_\ell - \sqrt{\frac{1-\delta+\delta\rho_r(y_\ell-y_r)^2}{1-\delta(\rho_E+\rho_d)}}, \text{ and} \tag{16}$$

$$\overline{x}_r = y_r + \sqrt{\frac{1-\delta+\delta\rho_\ell(y_\ell-y_r)^2}{1-\delta(\rho_E+\rho_d)}}. \tag{17}$$

Furthermore, (i) $\underline{x}$ and $\overline{x}$ are $\mathcal{C}^2$ on $(\underline{x}_\ell, y_\ell)\cup(y_\ell,y_r)\cup(y_r,\overline{x}_r)$; (ii) $\underline{x}(y_d)$ is strictly concave and $\overline{x}(y_d)$ strictly convex on each of those intervals; and (iii) $\frac{\underline{x}'(y_d)}{\overline{x}'(y_d)}$ is strictly decreasing over $y_d\in(y_\ell,y_r)$, with $\frac{\underline{x}'(y_d)}{\overline{x}'(y_d)}=1$ if and only if $y_d=\frac{\rho_\ell y_\ell+\rho_r y_r}{\rho_\ell+\rho_r}\in(y_\ell,y_r)$.

Proof. Direct computations yield $\underline{x}(y_d) = y_m - \phi(y_d)$ and $\overline{x}(y_d) = y_m + \phi(y_d)$ for each $y_d \in (\underline{x}_\ell, \overline{x}_r)$, where:

$$\phi(y_d) = \sqrt{\frac{1-\delta+\delta\rho_\ell(y_\ell-y_m)^2+\delta\rho_r(y_r-y_m)^2+\delta\rho_d(y_d-y_m)^2}{1-\delta\rho_E}}.$$

First, (16) follows from solving $y_d = \underline{x}(y_d)$ for $y_d < y_\ell$ and similarly (17) follows from solving $y_d = \overline{x}(y_d)$ for $y_d > y_r$. Next, $[\phi(y_d)]^2$ is a quadratic polynomial with a positive leading coefficient on each of the intervals $(\underline{x}_\ell, y_\ell)$, (y_ℓ, y_r), and $(y_r, \overline{x}_r)$, so it is strictly convex on each interval. Thus, on each interval $\underline{x}$ is strictly concave and $\overline{x}$ is strictly convex. Finally, direct computations yield that $\frac{\underline{x}'(y_d)}{\overline{x}'(y_d)}$ is strictly decreasing over $y_d \in (y_\ell, y_r)$, with $\frac{\underline{x}'(y_d)}{\overline{x}'(y_d)} = 1$ if and only if $y_d = \frac{\rho_\ell y_\ell+\rho_r y_r}{\rho_\ell+\rho_r}$. □

Lemma A7. *Under Assumptions 1 and 2, the mapping $\hat{y}_d(y_p) \equiv \operatorname{argmax}_{y_d\in[y_\ell,y_r]} U(y_d; y_p)$ is equivalent to a function $\hat{y}_d : X \to [y_\ell, y_r]$ that is continuous and weakly increasing. Furthermore, (i) $\hat{y}_d|_{[y_\ell,y_r]}$ has a unique fixed point y^*, (ii) $y_p < y^*$ implies $\hat{y}_d(y_p) \in (y_p, y^*]$, and (iii) $y_p > y^*$ implies $\hat{y}_d(y_p) \in [y^*, y_p)$.*

Proof. By Lemma A6, for all $y_d \in (y_\ell, y_r)$ we have: $\underline{x}''(y_d) < 0 < \underline{x}'(y_d)$ and $0 < \min\{\overline{x}'(y_d), \overline{x}''(y_d)\}$. Thus for an arbitrary $y_p \in X$ and $y_d \in (y_\ell, y_r)$, (i) $y_p \geq \overline{x}(y_d)$ implies $\frac{\partial U(y_d;y_p)}{\partial y_d} > 0$, (ii) $y_p \in (\underline{x}(y_d), \overline{x}(y_d))$ implies $\frac{\partial^2 U(y_d;y_p)}{\partial y_d^2} < 0$, and (iii) $y_p \leq \underline{x}(y_d)$ implies $\frac{\partial U(y_d;y_p)}{\partial y_d} < 0$, so $U(y_d; y_p)$ is strictly quasi-concave over $y_d \in [y_\ell, y_r]$ for all $y_p \in X$. Additionally, $\mu(y_d)$ is strictly increasing on $[y_\ell, y_r]$, so $U(y_d; y_p)$ satisfies the strict single-crossing condition on $[y_\ell, y_r] \times X$. Thus, $\hat{y}_d(y_p)$ is single-valued, continuous, and increasing. It follows that $\hat{y}_d|_{[y_\ell, y_r]}$ has a fixed point. To show it is unique, first note that: (i) $\hat{y}_d(y_\ell) = y_\ell$ if and only if $\left.\frac{\partial U(y_d;y_\ell)}{\partial y_d}\right|_{y_d=y_\ell^+} \leq 0$, (ii) $\hat{y}_d(y_r) = y_r$ if and only if $\left.\frac{\partial U(y_d;y_r)}{\partial y_d}\right|_{y_d=y_r^-} \geq 0$, and (iii) and $\hat{y}_d(y_p) = y_p \in (y_\ell, y_r)$ if and only if $\left.\frac{\partial U(y_d;y_p)}{\partial y_d}\right|_{y_d=y_p} = 0$. In the main text we define

$$\lambda(y_p) := \left.\frac{\partial U(y_d, y_p)}{\partial y_d}\right|_{y_d=y_p} = \rho_L \left.\frac{\partial \underline{x}(y_d)}{\partial y_d}\right|_{y_d=y_p} - \rho_R \left.\frac{\partial \overline{x}(y_d)}{\partial y_d}\right|_{y_d=y_p}.$$

An interior fixed point exists if and only if $\lambda(y_p) = 0$ for some $y_p \in (y_\ell, y_r)$. Strict concavity of $\underline{x}(y_d)$ and strict convexity of $\overline{x}(y_d)$ imply $\lambda'(y_p) < 0$. Therefore $\lambda(y_p) = 0$ at most once, which implies that y^* is unique. Furthermore, $y_p^* \in \{y_\ell, y_r\}$ if λ does not change sign, and otherwise $y^* \in (y_\ell, y_r)$. Finally, since $\hat{y}_d(y_p)$ weakly increasing and $\lambda'(y_p) < 0$, we know that (i) $y_p < y_p^*$ implies $\hat{y}_d(y_p) \in (y_p, y^*]$, and (ii) $y_p > y^*$ implies $\hat{y}_d(y_p) \in [y^*, y_p)$. □

Lemma A8. *Define* $\tilde{y}_d : [0, y_\ell] \to [\underline{x}_\ell, y_\ell]$ *as* $\tilde{y}_d(y_p) \equiv \operatorname{argmax}_{y_d \in [\underline{x}_\ell, y_\ell]} U(y_d, y_p)$. *Under Assumptions 1-2,* $\tilde{y}_d$ *is single-valued, continuous, increasing, and satisfies* $\tilde{y}_d(y_p) \in [\underline{\pi}, y_\ell]$ *for all* $y_p \leq y_\ell$. *Furthermore,* $\tilde{y}_d(y_\ell) = y_\ell$ *and otherwise* $\tilde{y}_d(y_p) \in (y_p, y_\ell)$. *A unique* $E_L < \underline{x}_\ell$ *exists such that (i)* $\tilde{y}_d(y_p) = \underline{x}_\ell$ *if* $y_p < E_L$, *(ii)* $\tilde{y}_d(y_p) > \underline{x}_\ell$ *if* $y_p > E_L$, *and (iii)* $\tilde{y}_d(y_p)$ *is strictly increasing on* (E_L, y_ℓ). *For* $y_p \geq y_r$, *analogous properties hold for* $\operatorname{argmax}_{y_d \in [y_r, \overline{x}_r]} U(y_d, y_p)$.

Proof. Fix $y_p \leq y_\ell$. To begin, we show that $\tilde{y}_d(y_p) \subset [\underline{\pi}, y_\ell]$. The result is trivial if $\underline{\pi} = \underline{x}_\ell$. If $\underline{\pi} > \underline{x}_\ell$, then an interval $Z \subseteq [\underline{x}_\ell, \underline{\pi}]$ exists where $\mu(y_d)$ is decreasing in y_d. Moreover, $\rho_R > 1/2\delta$ by Lemma 4. Since $A(y_d)$ is continuously differentiable on M_L under Assumption 2 at rates of change that decrease continuously in magnitude by Lemma A6, it follows that $\mu(y_d)$ is strictly decreasing on $[\underline{x}_\ell, \underline{\pi})$. Thus Lemma 4 implies $U(y_d, y_p)$ is strictly increasing in y_d on $[\underline{x}_\ell, \underline{\pi})$ for all $y_p \leq y_\ell$ so $\tilde{y}_d(y_p) \subset [\underline{\pi}, y_\ell]$.

Since $\mu(y_d)$ strictly increases on $[\underline{\pi}, y_\ell]$, it follows that $U(y_d, y_p)$ satisfies strict single crossing. Thus every selection from $\tilde{y}_d(y_p)$ must be increasing, in addition to $\tilde{y}_d(y_p)$ being non-empty, upper hemicontinuous, and compact-valued by Berge's maximum theorem. Furthermore, Lemma A1 implies that $\tilde{y}_d(y_\ell) = y_\ell$ and otherwise $\tilde{y}_d(y_p) \subset (y_p, y_\ell)$. Thus, there is a unique

$E_L \equiv \inf\{y_p < \underline{x}_\ell : \tilde{y}_d(y_p) \subset (\underline{x}_\ell, y_\ell)\}$. For all $y_p \in (E_L, y_\ell)$, any $y_d \in \tilde{y}_d(y_p)$ must satisfy

$$\frac{\partial U(y_d; y_p)}{\partial y_d} \propto -\rho_d(y_d - y_p) - \underline{x}'(y_d)[\rho_L(\underline{x}(y_d) - y_p) - \rho_R(\overline{x}(y_d) - y_p)] = 0 \tag{18}$$

and

$$\frac{\partial^2 U(y_d; y_p)}{\partial y_d^2} \propto -\rho_d - \underline{x}''(y_d)[\rho_L(\underline{x}(y_d) - y_p) - \rho_R(\overline{x}(y_d) - y_p)] - [\underline{x}'(y_d)]^2 \rho_E < 0. \tag{19}$$

Since $y_p < \tilde{y}_d(y_p)$ and $\underline{x}'(y_d) > 0$, (18) holds only if $\rho_L(\underline{x}(y_d) - y_p) - \rho_R(\overline{x}(y_d) - y_p) < 0$.

To show $\tilde{y}_d(y_p)$ is single-valued for all $y_p \in (E_L, y_\ell)$, suppose not and let $y_d, y_d' \in \tilde{y}_d(y_p)$ where $y_d < y_d'$. Then, there must be a $y \in (y_d, y_d')$ satisfying $\rho_L(\underline{x}(y) - y_p) - \rho_R(\overline{x}(y) - y_p)) = -\left(\frac{\rho_d + \rho_E[\underline{x}'(y)]^2}{\underline{x}''(y)}\right) > 0$. And since $\frac{\partial}{\partial y}[\rho_L(\underline{x}(y) - y_p) - \rho_R(\overline{x}(y) - y_p)] = \underline{x}'(y)\rho_E > 0$, we must also have $\rho_L(\underline{x}(y_d') - y_p) - \rho_R(\overline{x}(y_d') - y_p) > 0$. But then (18) fails at y_d' which implies that $y_d' \notin \tilde{y}_d(y_p)$, contradicting our assumption that $y_d' \in \tilde{y}_d(y_p)$. Consequently, $\tilde{y}_d(y_p)$ is single-valued. Finally, applying the implicit function theorem to $\tilde{y}_d(y_p)$ shows that it is strictly increasing on $[E_L, y_p]$. Analogous arguments establish the result for $y_p \geq y_r$. □

C.1 Proof of Proposition 2.

First, Lemma A1 implies that any fixed point of $y_d^*|_{(E_L,E_R)}$ must be in $[y_\ell, y_r]$, where $y_d^*(y_p) = \hat{y}_d(y_p)$. Second, by Lemma A7: $y_d^*|_{[y_\ell,y_r]}$ has a unique fixed point; $y_p < y^*$ implies $\hat{y}_d(y_p) \in (y_p, y^*]$; and $y_p > y^*$ implies $\hat{y}_d(y_p) \in [y^*, y_p)$. Third, by Proposition 1: $y_p \in (E_L, \underline{y}_p)$ implies $y_d^*(y_p) \in (y_p, y_\ell)$; $y_p \in (\underline{y}_p, \overline{y}_p)$ implies $y_d^*(y_p) = \hat{y}_d(y_p)$; and $y_p \in (\overline{y}_p, E_R)$ implies $y_d^*(y_p) \in (y_r, y_p)$. Thus, y^* is the unique fixed point of $y_d^*|_{(E_L,E_R)}$. □

C.2 Proof of Proposition 3.

By Lemma A1, (i) $y_p < y_\ell$ implies $y_\ell \notin y_d^*(y_p)$ and (ii) $y_p > y_r$ implies $y_r \notin y_d^*(y_p)$. Then, since $y_d^*(y_p)$ is increasing, (i) $\underline{y}_p = y_\ell$ if and only if $y^* = y_\ell$ and (ii) $\overline{y}_p = y_r$ if and only if $y^* = y_\ell$. Therefore uniqueness of y^* implies $y_r \in \Delta$ or $y_\ell \in \Delta$. The characterization using λ follows directly from the characterization of y^* in Lemma A7. □

C.3 Proof of Proposition 4.

Fix $\rho_E \equiv \rho_L + \rho_R$. Thus, when we refer to increasing ρ_L throughout the proof, we are implicitly decreasing ρ_R by the same amount. Before proceeding, note that since ρ_E is constant, $A(y_d)$ is constant. Therefore $\frac{\partial \lambda(y_d)}{\partial \rho_L} - \frac{\partial \lambda(y_d)}{\partial \rho_R} = \frac{\partial \underline{x}(y_d)}{\partial y_d} + \frac{\partial \overline{x}(y_d)}{\partial y_d} > 0$ for all $y_d \in (y_\ell, y_r)$.

1. Since $\frac{\partial \lambda(y_d)}{\partial \rho_L} - \frac{\partial \lambda(y_d)}{\partial \rho_R} > 0$ for all $y_d \in (y_\ell, y_r)$, we know (i) $\lambda(y_\ell) \leq 0$ implies $y^* = y_\ell$, (ii) $\lambda(y_r) \geq 0$ implies $y^* = y_r$, and (iii) otherwise $\lambda(y^*) = 0$ at $y^* \in (y_\ell, y_r)$. Thus, y^* is weakly increasing.

2. From Lemma A8, E_L is the smallest $y_p < \underline{x}_\ell$ such that $\frac{\partial U(y_d, E_L)}{\partial y_d}\Big|_{y_d = \underline{x}_\ell^+} \geq 0$. Then, computation yields $\frac{\partial^2 U(y_d, E_L)}{\partial y_d \partial \rho_L}\Big|_{y_d = \underline{x}_\ell^+} - \frac{\partial^2 U(y_d, E_L)}{\partial y_d \partial \rho_R}\Big|_{y_d = \underline{x}_\ell^+} = -\frac{\delta \rho_d[(\underline{x}_\ell - E_L) + (\overline{x}_r - E_L)]}{1 - \delta \rho_E} < 0$, so E_L weakly increases in ρ_L. By an analogous argument, E_R weakly increases in ρ_L.

   □

D Fixed Median

D.1 Proof of Proposition 5.

From the main text, $y_d^*(y_p)\Big|_{(E_L, E_R)} = (1 - \delta \rho_E) y_p + \delta \rho_E y_M$. Applying the envelope theorem yields $\nu'(y_p)\Big|_{(E_L, \underline{x}_\ell)} = \rho_d(y_d^*(y_p) - \underline{x}_\ell) > 0$ and $\nu'(y_p)\Big|_{(\underline{x}_\ell, y_\ell)} = \frac{-(\delta \rho_E)^2 \rho_d (y_m - y_p)}{1 - \delta \rho_E} < 0$. Thus, the result follows from continuity of ν. Analogously, ν strictly increases on $[y_m, \overline{x}_r]$ and strictly decreases on $[\overline{x}_r, E_R]$. □

D.2 Proof of Lemma 5.

Direct computations show $\frac{\partial^2 \underline{x}(y_a, y_b)}{\partial y_a \partial y_b} < 0$. Using this fact, it is straightforward to sign $\frac{\partial y_a(y_b)}{\partial y_b}$ by applying the implicit function theorem to (11). The result for $y_b(y_a)$ is analogous. □

D.3 Proof of Proposition 6.

Proposition 1 implies $y_a^* \in (y_{p_a}, y_M)$ and $y_b^* \in (y_M, y_{p_b})$. Lemma 5 implies uniqueness. □

E Mass Representation

E.1 Proof of Proposition 7.

It follows from Milgrom and Shannon (1994) and Cho and Duggan (2003) that it is sufficient to show that $U(y_d, y_p)$ satisfies the single-crossing property on X. By Lemma 4, $\max\{1 - \tilde{P}(y_\ell), \tilde{P}(y_\ell)\} \leq \frac{1}{2\delta}$ implies $\mu(y_d)$ is increasing on X. Thus $U(y_d, y_p)$ satisfies increasing differences on X, which implies single crossing. □

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